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Approximate Optimal Control Governed by some Parabolic Equations via Laguerre Polynomials Collocation Approach

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ABSTRACT

The present paper proposes a novel numerical approach for approximating solutions to optimal control problems with parabolic constraints. Utilizing Laguerre polynomials as a novel basis set, a method was developed to address a class of this problem. The employment of these basis functions in conjunction with the collocation method facilitates the transformation of optimal control problems governed by parabolic constraints into a system of nonlinear algebraic equations. The present study proposes an efficient discretization and transformation of complex optimal control problems governed by parabolic equations into lower-dimensional algebraic systems by leveraging the unique properties of Laguerre polynomials. Convergence analysis has been demonstrated to ascertain the optimal value approximations of the proposed method. In order to provide a comprehensive illustration of the reliability and applicability of the proposed method, two illustrative examples are presented. The findings underscore the efficacy and precision of the implemented methodology. This work makes a significant contribution to the field by offering a robust framework for solving complex parabolic control problems, thereby demonstrating the potential of spectral methods in the context of optimal control theory.

I. Introduction

Recent advances in the theory and numerical analysis of optimal control problems governed by parabolic partial differential equations have significantly enhanced the capacity to model and solve complex systems in science and engineering. Parabolic partial differential equations, including the heat equation, are instrumental in the simulation of dynamic processes, such as the distribution of temperature, diffusion phenomena, and energy transport. The utilization of these equations is pervasive, encompassing diverse disciplines such as physics, biology, and engineering. Their application lies in the comprehension of imprecise dynamics, chaotic systems, and tangible phenomena in the real world, including fluid flow and chemical reactions [1, 8]. The optimal control of such systems entails the formulation of strategies that enhance performance while adhering to the constraints imposed by the governing equations and boundary conditions.

The study of optimal control problems for parabolic equations has introduced innovative approaches to addressing challenges related to the minimization or maximization of cost functionals. For instance, Casas [1] investigated semilinear parabolic equations with memory effects, deriving first and second order necessary optimality conditions and analyzing associated inverse systems. Similarly, Na [8] examined degenerate parabolic equations, highlighting their applications in economics and physical modeling. These studies emphasize theoretical advancements in understanding the behavior of parabolic systems within the framework of optimal control. Führer proposed a space-time least-squares finite element method for distributed optimal control problems [2]. In addition, in [12] employed algorithms based on the Pontryagin maximum principle for semilinear parabolic equations and demonstrated their convergence and efficiency in solving such problems. Other approaches, such as the Ritz method combined with Legendre polynomial bases, were presented by Mamehrashi et al. in [6]. Similarly, Hosseini and co-authors utilized block-pulse Legendre functions to solve optimal control problems subject to parabolic differential equation constraints [4]. Latifi [5] applied the Jacobi-Gauss-Radau Lagrangian method, discretizing variables to create a parametric framework for solving parThe paucityabolic optimal control problems. In [9] modified variational iteration method (MVIM) to solve nonlinear optimal control problems more efficiently than the standard VIM by eliminating extra computations. The approach first transforms the optimal control problem into a two-point boundary value problem using Pontryagin's maximum principle and then applies MVIM with a Taylor series expansion for nonlinear terms. For additional numerical approaches see [10, 11].

The paucity of analytical solutions for optimal control problems with parabolic constraints (OCPPC) necessitates the development of robust numerical and approximate methods. This paper introduces a novel approach utilizing Laguerre polynomials, which represents a significant advancement in addressing these complex challenges. By leveraging a spectral method based on collocation points, this technique reduces computational complexity and enhances the accuracy of approximating solutions. Moreover, it has been demonstrated to expand analytical and practical capabilities, providing a comprehensive framework for tackling optimal control problems. The innovative application of Laguerre polynomials in this context signifies a pioneering effort within the field, addressing a critical gap in the extant literature and proposing novel avenues for research and application in optimal control theory. The objective of this work is to make a substantial contribution to the advancement of knowledge in the domain of parabolic optimal control problems, thereby laying the foundation for future progress in this field

The present study proposes an efficient discretization and transformation of complex optimal control problems governed by parabolic equations into lower-dimensional algebraic systems by leveraging the unique properties of Laguerre polynomials. This approach not only enhances the performance and accuracy of existing numerical methods but also opens promising avenues for future research in optimal control. The present study focuses on minimizing a cost functional that is dependent on the state variable $z(x,t)$, and the control input $u(x,t)$. These variables are governed by a partial differential equation (PDE) involving second-order spatial derivatives, as well as the direct influence of the control input. The cost functional, which has been defined, serves as a critical component in formulating the optimal control problem. This ensures that the proposed method effectively addresses the challenges posed by parabolic constraints. The following structure has been employed in the composition of the present manuscript: In Section II, the mathematical formulation and problem setup are presented. The Section III of this text is devoted to a discussion of the Laguerre polynomial approximation method and its application to optimal control. The convergence analysis is substantiated in Section IV. In Section V, the numerical implementation and experimental results are described, and a discussion of the results is provided. Finally, the conclusion is stated in Section VI.

II. Methodology

A. Problem Statement

We consider a class of optimal control problems governed by parabolic partial differential equations. The aim is to determine the state and control functions $z(x,t)$ and $u(x,t)$, respectively, that minimize the following quadratic cost functional [6]:

$$J(z,u) = \frac{1}{2}\int_0^1 \int_0^R x^k[Az^2(x,t) + Bu^2(x,t)]dx\,dt; \qquad (1)$$

subject to the following dynamic constraints:

$$\frac{\partial z(x,t)}{\partial t} = \beta\left(\frac{\partial^2 z(x,t)}{\partial x^2} + \frac{k}{x}\frac{\partial z(x,t)}{\partial x}\right) + u(x,t); \qquad (2)$$

and the initial and boundary condition:

$$z(x,0) = z_0(x), \quad 0 < x < R, \quad z(R,t) = 0, \quad t > 0. \qquad (3)$$

That A and B are defined as two arbitrary functions, β is a positive constant representing the diffusivity coefficient, R is characterized as a positive real number, and k may assume the values of 1 or 2. Also the functions $z(x,t)$ and $u(x,t)$ are differentiable and exhibit smooth characteristics. The aim is to solve the above problem with the Laguerre polynomials, which is defined below.

B.Definition and Properties of 2D Laguerre Polynomials

The two-dimensional Laguerre polynomials $L_{m,n}(z,z')$, defined for two (generally independent) complex variables z and z', constitute a powerful basis for representing multivariate functions. These polynomials are employed for approximate solutions to a wide range of problems across various domains. The 2D Laguerre polynomials, are defined as follows [13, 14, 15, 16, 17]:

$$L_{m,n}(z,z') \equiv \exp\left(-\frac{\partial^2}{\partial z \partial z'}\right) z^m z'^n. \qquad (4)$$

Expanding the exponential operator yields the explicit polynomial form:

$$L_{m,n}(z,z') = \sum_{j=0}^{\min\{m,n\}} \frac{(-1)^j m!n!}{j!(m-j)!(n-j)!} z^{m-j} z'^{n-j}. \qquad (5)$$

The inverse of relation (4) is computed as follow

$$z^m z'^n = \exp\left(\frac{\partial^2}{\partial z \partial z'}\right) L_{m,n}(z,z')$$

$$= \sum_{j=0}^{\min\{m,n\}} \frac{m!n!}{j!(m-j)!(n-j)!} L_{m-j,n-j}(z,z'). \qquad (6)$$

Remark 1. The following Laguerre polynomials are directly related to the inversion (6) [17].

$$\begin{aligned} L_{m,n}(z,0) &= \frac{(-1)^n m!}{(m-n)!} z^{m-n}; \\ L_{m,n}(0,z') &= \frac{(-1)^m n!}{(n-m)!} z'^{n-m}; \\ L_{m,n}(0,0) &= (-1)^n n!\, \delta_{m,n}; \\ L_{m,0}(z,z') &= z^m, \quad L_{0,n}(z,z') = z'^n, \quad L_{0,0}(z,z') = 1. \end{aligned}$$

Remark 2. We have the following relations for the partial derivatives of the Laguerre polynomials [17].

$$\begin{aligned} \frac{\partial}{\partial z} L_{m,n}(z,z') &= mL_{m-1,n}(z,z'); \\ \frac{\partial}{\partial z'} L_{m,n}(z,z') &= nL_{m,n-1}(z,z'). \end{aligned}$$

These relations are instrumental in deriving system dynamics and adjoint equations in the Laguerre spectral domain. They also satisfy the following recurrence identities [17]:

$$\begin{aligned} L_{m+1,n}(z,z') &= zL_{m,n}(z,z') - nL_{m,n-1}(z,z'); \\ L_{m,n+1}(z,z') &= z'L_{m,n}(z,z') - mL_{m-1,n}(z,z'). \end{aligned} \tag{7}$$

Since the problems considered in this manuscript are based on the real values x and t, all the relations stated in section 3 hold accordingly. In particular, the recurrence relation (7) takes the following form:

$$\begin{aligned} L_{m+1,n}(x,t) &= xL_{m,n}(x,t) - nL_{m,n-1}(x,t); \\ L_{m,n+1}(x,t) &= tL_{m,n}(x,t) - mL_{m-1,n}(x,t). \end{aligned} \tag{8}$$

III. Numerical solution method

In this section, we provide an approximate solution for problem (1)–(3). For this

$$\begin{cases} z(x,t) & \cong z_{M,N}(x,t) = \sum_{i=0}^{M} \sum_{j=0}^{N} c_{ij} L_{ij}(x,t); \\ u(x,t) & \cong u_{M,N}(x,t) = \sum_{i=0}^{M} \sum_{j=0}^{N} d_{ij} L_{ij}(x,t); \\ & 0 \le x \le x_f, \quad 0 \le t \le t_f. \end{cases} \tag{9}$$

We can rewrite the solution function (9) in the following matrix form

$$z_{M,N}(x,t) = \sum_{i=0}^{M} \sum_{j=0}^{N} c_{ij} L_{ij}(x,t) = \mathrm{C}^T \mathrm{L}(x,t); \tag{10}$$

that C is an $(M+1) \times (N+1)$ matrix whose entries are real-valued coefficients c_{ij}, where $i = 0,1,\dots,M$ and $j = 0,1,\dots,N$ and the matrix form are defined as follows:

$$\mathrm{C} = \begin{bmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,N} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ c_{M,0} & c_{M,1} & \cdots & c_{M,N} \end{bmatrix},$$

c_{ij} denotes the unknown coefficients and will be determined during the solving process. L(x,t) represents a $(M+1) \times (N+1)$ matrix, whose elements are Laguerre polynomials $L_{ij}(x,t)$, where $i = 0,1,\dots,M$ and $j = 0,1,\dots,N$.

$$\mathrm{L}(x,t) = \begin{bmatrix} L_{0,0}(x,t) & L_{0,1}(x,t) & \cdots & L_{0,N}(x,t) \\ L_{1,0}(x,t) & L_{1,1}(x,t) & \cdots & L_{1,N}(x,t) \\ \vdots & \vdots & \ddots & \vdots \\ L_{M,0}(x,t) & L_{M,1}(x,t) & \cdots & L_{M,N}(x,t) \end{bmatrix}.$$

The aforementioned 3×3 matrix form is as follows:

$$\mathrm{L}(x,t) = \begin{bmatrix} 1 & t & t^2 \\ x & -1 + tx & -2t + t^2 x \\ x^2 & -2x + tx^2 & 2 - 4tx + t^2 x^2 \end{bmatrix}$$

*A. Partial derivative with respect to **t** and **x**(First-Order)*

The first-order partial derivative with respect to t is given by:

$$\frac{\partial z(x,t)}{\partial t} = \mathrm{C}^T \frac{\partial \mathbf{L}(x,t)}{\partial t}. \tag{11}$$

Here

$$\frac{\partial \mathrm{L}(x,t)}{\partial t} = \begin{bmatrix} \frac{\partial L_{0,0}(x,t)}{\partial t} & \frac{\partial L_{0,1}(x,t)}{\partial t} & \cdots & \frac{\partial L_{0,N}(x,t)}{\partial t} \\ \frac{\partial L_{1,0}(x,t)}{\partial t} & \frac{\partial L_{1,1}(x,t)}{\partial t} & \cdots & \frac{\partial L_{1,N}(x,t)}{\partial t} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial L_{M,0}(x,t)}{\partial t} & \frac{\partial L_{M,1}(x,t)}{\partial t} & \cdots & \frac{\partial L_{M,N}(x,t)}{\partial t} \end{bmatrix}.$$

where $\frac{\partial \mathrm{L}(x,t)}{\partial t}$ denotes a $(M+1) \times (N+1)$ matrix, whose elements are given by $\frac{\partial L_{ij}(x,t)}{\partial t}$, where $i = 0,1,\dots,M$ and $j = 0,1,\dots,N$. Using the properties of two-dimensional Laguerre polynomials

$$\frac{\partial}{\partial t} L_{m,n}(x,t) = nL_{m,n-1}(x,t).$$

Thus

$$\frac{\partial z(x,t)}{\partial t} = \mathrm{C}^T \mathrm{L}_t(x,t),$$

where

$$\mathrm{L}_t(x,t) = \left[nL_{m,n-1}(x,t) \right]_{m=0,\dots,M, n=1,\dots,N}.$$

The first-order partial derivative with respect to x is given by,

$$\frac{\partial z(x,t)}{\partial x} = \mathrm{C}^T \frac{\partial \mathrm{L}(x,t)}{\partial x}. \tag{12}$$

Here

$$\frac{\partial \mathrm{L}(x,t)}{\partial x} = \begin{bmatrix} \frac{\partial L_{0,0}(x,t)}{\partial x} & \frac{\partial L_{0,1}(x,t)}{\partial x} & \cdots & \frac{\partial L_{0,N}(x,t)}{\partial x} \\ \frac{\partial L_{1,0}(x,t)}{\partial x} & \frac{\partial L_{1,1}(x,t)}{\partial x} & \cdots & \frac{\partial L_{1,N}(x,t)}{\partial x} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial L_{M,0}(x,t)}{\partial x} & \frac{\partial L_{M,1}(x,t)}{\partial x} & \cdots & \frac{\partial L_{M,N}(x,t)}{\partial x} \end{bmatrix}.$$

where $\frac{\partial \mathrm{L}(x,t)}{\partial x}$ denotes a $(M+1) \times (N+1)$ matrix, whose elements are given by $\frac{\partial L_{ij}(x,t)}{\partial x}$, where $i = 0,1,\dots,M$ and $j = 0,1,\dots,N$. Using the properties of two-dimensional Laguerre polynomials

$$\frac{\partial}{\partial x} L_{m,n}(x,t) = mL_{m-1,n}(x,t).$$

Thus

$$\frac{\partial z(x,t)}{\partial x} = \mathrm{C}^T \mathrm{L}_x(x,t);$$

where:

$$\mathrm{L}_x(x,t) = \left[mL_{m-1,n}(x,t)\right]_{m=1,\dots,M, n=0,\dots,N}.$$

B. Partial derivative with respect to $\boldsymbol{x}$ *(Second-Order)*

The second-order partial derivative with respect to x is given by

$$\frac{\partial^2 z(x,t)}{\partial x^2} = \mathrm{C}^T \frac{\partial^2 \mathrm{L}(x,t)}{\partial x^2}.$$

Here

$$\frac{\partial^2 \mathrm{L}(x,t)}{\partial x^2} = \begin{bmatrix} \frac{\partial^2 L_{0,0}(x,t)}{\partial x^2} & \frac{\partial^2 L_{0,1}(x,t)}{\partial x^2} & \dots & \frac{\partial^2 L_{0,N}(x,t)}{\partial x^2} \\ \frac{\partial^2 L_{1,0}(x,t)}{\partial x^2} & \frac{\partial^2 L_{1,1}(x,t)}{\partial x^2} & \dots & \frac{\partial^2 L_{1,N}(x,t)}{\partial x^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 L_{M,0}(x,t)}{\partial x^2} & \frac{\partial^2 L_{M,1}(x,t)}{\partial x^2} & \dots & \frac{\partial^2 L_{M,N}(x,t)}{\partial x^2} \end{bmatrix}.$$

where $\frac{\partial^2 \mathbf{L}(x,t)}{\partial x^2}$ denotes a $(M+1)\times(N+1)$ matrix, whose elements are given by $\frac{\partial^2 L_{ij}(x,t)}{\partial x^2}$, where $i = 0,1,\dots,M$ and$j = 0,1,\dots,N$. Using the properties of two-dimensional Laguerre polynomials

$$\mathrm{L}_{xx}(x,t) = \frac{\partial^2}{\partial x^2} L_{m,n}(x,t) = m(m-1)L_{m-2,n}(x,t);$$
$$\text{if} \quad m \geq 2, \quad n \geq 0.$$

Thus

$$\frac{\partial^2 z(x,t)}{\partial x^2} = \mathrm{C}^T \mathrm{L}_{xx}(x,t).$$

Let us now assume that

$$u_{M,N}(x,t) = \sum_{i=0}^{M} \sum_{j=0}^{N} d_{ij} L_{ij}(x,t) = \mathrm{D}^T \mathrm{L}(x,t); \quad (13)$$

that D is an $(M+1)\times(N+1)$ matrix whose entries are real-valued coefficients d_{ij}, where $i = 0,1,\dots,M$ and $j = 0,1,\dots,N$ and the matrix form are defined as follows:

$$\mathrm{D} = \begin{bmatrix} d_{0,0} & d_{0,1} & \dots & d_{0,N} \\ d_{1,0} & d_{1,1} & \dots & d_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ d_{M,0} & d_{M,1} & \dots & d_{M,N} \end{bmatrix}.$$

d_{ij} denotes the unknown coefficients and will be determined during the solving process. Taking into account for the dynamical system (2), we have

$$\mathrm{C}^T \mathrm{L}_t(x,t) - \beta(\mathrm{C}^T \mathrm{L}_{xx}(x,t) + \frac{k}{x} \mathrm{C}^T \mathrm{L}_x(x,t)) -$$
$$\mathrm{D}^T \mathrm{L}(x,t) \simeq \hat{G}(x,t) \simeq 0. \quad (14)$$

C. Selection of Collocation Points

This discretization transforms the differential equations into a system of algebraic equations. Furthermore, by taking the collocation points as

$$x_i = x_0 + \frac{x_f - x_0}{2(M+1)(N+1)-2} i; \quad i = 1,\dots,M,$$

$$t_j = t_0 + \frac{t_f - t_0}{2(M+1)(N+1)-2} j; \quad j = 1,\dots,N,$$

and substituting them into Eq. (14), it leads to the following system of algebraic equations

$$\Lambda_{ij} \cong \tilde{G}(x_i, t_j) = 0; \quad i = 1,\dots,M, \quad j = 1,\dots,N. \quad (15)$$

The initial condition from Eq. (3) can be rewritten as:

$$\Lambda_{0,0} \cong C^T \mathrm{L}(x_0, 0) - z_0(x) = 0. \quad (16)$$

The cost function, as defined in Eq. (1), can be approximated as follows:

$$J(\mathrm{Z},\mathrm{U}) = \frac{1}{2}\int_0^1 \int_0^R x^k [A(\mathrm{C}^T \mathrm{L}(x,t))^2 +$$
$$B(\mathrm{D}^T \mathrm{L}(x,t))^2]\, dx\, dt.$$

To enforce the constraints and optimize the performance index, the Lagrange multipliers method is employed. The augmented functional is defined as

$$J^*(\mathrm{Z},\mathrm{U},\lambda) = J(\mathrm{Z},\mathrm{U}) + \sum_{i=0}^{M} \sum_{j=0}^{N} \lambda_{ij} \Lambda_{ij};$$

where λ_{ij} are the Lagrange multipliers corresponding to the residuals. The necessary conditions for optimality are

$$\frac{\partial J^*}{\partial \mathrm{Z}} = 0; \quad \frac{\partial J^*}{\partial \mathrm{U}} = 0; \quad \frac{\partial J^*}{\partial \lambda} = 0. \quad (17)$$

To solve (17), various techniques have been developed for addressing nonlinear optimization problems. In this study, we have opted to utilize the techniques provided by the available software tools.

IV. Convergence Analysis

In this section, the convergence of the method presented in Section 4 is examined.

Theorem .۱ For each $\Omega \in C^2([0,x_f]\times[0,t_f])$, there exists a sequence of polynomials $L_{ij}(x,t)\} \in \Omega$ that converges uniformly to $\hat{z}(x,t)$.

Proof. Refer to [3] for the proof.

Lemma 1. Let $\Omega \subset C^2([0,R]\times[0,1])$ denote the set of admissible pairs (z,u) satisfying the following parabolic partial

$$\frac{\partial z(x,t)}{\partial t} = \beta\left(\frac{\partial^2 z}{\partial x^2} + \frac{k}{x}\frac{\partial z}{\partial x}\right) + u(x,t);$$

with the initial and boundary conditions:

$$z(x,0) = z_0(x), \quad 0 < x < R,$$
$$z(R,t) = 0, \quad 0 < t < 1. \quad (18)$$

The associated cost functional is defined by

$$J(z,u) = \frac{1}{2}\int_0^1 \int_0^R x^k [Az^2(x,t) + Bu^2(x,t)]dx\, dt.$$

Let $\{\Omega_{MN}\}_{M,N=0}^{\infty}$ as a sequence of subspaces spanned by Laguerre polynomials of degree at most MN. Define the infimum of the cost functional J over each subspace as

$$\gamma_{MN} = \inf_{(z,u)\in\Omega_{MN}} J(z,u);$$

then,

$$\lim_{M,N\to\infty} \gamma_{MN} = \inf_{(z,u)\in\Omega} J(z,u).$$

Proof. Consider the cost functional $J{:}\,\Omega \to \mathbb{R}$ defined by

$$J(z,u) = \frac{1}{2}\int_0^1 \int_0^R x^k[Az^2(x,t) + Bu^2(x,t)]dx\,dt. \qquad (19)$$

where $(z,u) \in \Omega$ satisfy in (۱۸) and (19). Let $\{L_{ij}(x,t)\}_{i,j=0}^{\infty}$ denote the Laguerre basis on $[0,R] \times [0,1]$. Define Ω_{MN} as the finite-dimensional subspace consisting of pairs (z_{MN}, u_{MN}) expressed as

$$z_{MN}(x,t) = \sum_{i=0}^{M} \sum_{j=0}^{N} c_{ij} L_{ij}(x,t); \qquad (20)$$

$$u_{MN}(x,t) = \sum_{i=0}^{M} \sum_{j=0}^{N} d_{ij} L_{ij}(x,t). \qquad (21)$$

where the coeffcients $\{c_{ij}, d_{ij}\}$ are unknown and will be determined during the solving process.

Define the infimum of the cost functional over each subspace Ω_{MN} as

$$\gamma_{MN} = \inf_{(z,u)\in\Omega_{MN}} J(z,u).$$

That the set Ω_{MN} is finite-dimensional and consists of continuous functions that meet theinitial and boundary conditions. Since J is continuous on Ω, we conclude that γ_{MN} is well-defined. Since every subspace of Laguerre polynomials of degree up to MN is contained within the subspace of Laguerre polynomials of degree up to $(M+1)(N+1)$, it follows that $\Omega_{MN} \subset \Omega_{(M+1)(N+1)}$.

Since J is defined as an infimum over each subspace Ω_{MN}, we have

$$\gamma_{(M+1)(N+1)} = \inf_{(z,u)\in\Omega_{(M+1)(N+1)}} J(z,u) \leq \inf_{(z,u)\in\Omega_{MN}} J(z,u) = \gamma_{MN}.$$

Consequently,

$$\gamma_{(M+1)(N+1)} \leq \gamma_{MN}.$$

Thus the sequence $\{\gamma_{MN}\}$ is monotonically decreasing

$$\gamma_{MN} \leq \gamma_{(M-1)(N-1)} \leq \cdots \leq \gamma_{11}.$$

Each subspace $\Omega_{MN} \subset \Omega$ consists of functions (z,u) that satisfy the same initial and boundary conditions as those in Ω. Consequently, the restriction of J to $\Omega_{N,M}$ is also bounded below by $\inf_{\Omega} J$. Specifically, we have

$$\gamma_{MN} \geq \inf_{\Omega}.$$

By the monotone convergence theorem, since $\{\gamma_{MN}\}$ is monotonically decreasing and bounded below, the limit exists. Define this limit as

$$\lim_{M,N\to\infty} \gamma_{MN} = L;$$

where $L \geq \inf_{\Omega} J$. To show that $L = \inf_{\Omega} J$, let $\epsilon > 0$ be arbitrary. By the definition of $\inf_{\Omega} J$, there exists a function $(z,u) \in \Omega$ such that

$$J(z,u) < \inf_{\Omega} J + \epsilon.$$

Since Laguerre polynomials are dense in $C^2([0,x_f] \times [0,t_f])$, we can approximate (z,u) by a sequence $(z_{MN}, u_{MN}) \in \Omega_{MN}$ that converges uniformly to (z,u). By continuity of J, we find

$$\lim_{M,N\to\infty} J(z_{MN}, u_{MN}) = J(z^*, u^*);$$

implying that

$$L < \inf_{\Omega} J + \epsilon.$$

Since ϵ is arbitrary, it follows that $L = \inf_{\Omega} J$.

Thus, we conclude that

$$\lim_{M,N\to\infty} \gamma_{MN} = \inf_{\Omega} J\,.$$

V. Numerical Implementation and Results

This section includes two numerical examples to demonstrate the effectiveness and computational efficiency of the proposed method. The findings substantiate the dependability and efficacy of the Laguerre-based methodology. The simulated results have been carried out using Mathematica 11 on a 2.53 MHz Alpha Machin with 8GB RAM.

Example 1. Consider the following OCPPC:

$$\min J(z,u) = \frac{1}{2}\int_0^1 \int_0^1 x[z^2(x,t) + u^2(x,t)]\,dx\,dt;$$

subject to the constraint

$$\frac{\partial z(x,t)}{\partial t} = \frac{\partial^2 z(x,t)}{\partial x^2} + \frac{1}{x}\frac{\partial z(x,t)}{\partial x} + u(x,t);$$

with the following initial and boundary conditions:

$$z(x,0) = 1 - x^2, \quad 0 < x < 1, \quad z(1,t) = 0; \quad t > 0. \quad (22)$$

This example is numerically solved by the proposed method for kind of iteration; $M = 2, N = 2$; $M = 2, N = 4$; $M = 3, N = 7$ and $M = 4, N = 4$. A comparison of the cost function values across different iterations for the proposed method and the methods presented in [4] and [6] is provided in Table 1. As shown in Table 1, for example, in iterations $M = 3, N = 7$, the cost functional value obtained by our method is 0.00292532, while for the method proposed in [6] is 0.013027. This demonstrates that our approach yields better results in comparable iterations. Furthermore, when examining the cost function values overall, our method provides superior results even in iterations $M = 2, N = 2$, outperforming the methods proposed in [4] and [6]. Figure 1 shows the state function $z(x,t)$ over the space-time domain $[0,1]\times[0,1]$ for $M=2, N=2$. The plotted surface is smooth and continuous, clearly reflecting the system dynamics under the influence of optimal control. This observation indicates that the proposed method yields qualitatively accurate results even at a low order of approximation with $M=2$ and $N=2$. The corresponding optimal control function $u(x,t)$ is presented in Figure 2. The control surface is smooth and well-behaved, exhibiting no signs of numerical irregularities such as oscillations or instability. The relatively small magnitude of the control values suggests that the obtained solution is not only stable but also effective in minimizing the control effort, which is a fundamental goal in optimal control problems. The behavior of the state function $z(t)$ at a fixed spatial point $x=0.2$ for three different iterations designed in Figure 3 for $M=3, N=7$, $M=3, N=3$, and $M=4, N=6$. As depicted in the figure, the solution becomes progressively smoother and more accurate with increasing values of the approximation parameters M and N, demonstrating the convergence and improved performance of the proposed numerical method. Figure 4 shows the behavior of the control function $u(t)$ evaluated at $x=0.2$. As can be seen, the control profile $u_{46}(t)$ captures more precise variations compared to $u_{33}(t)$ and $u_{24}(t)$, indicating that higher-order approximations provide a more accurate representation of the control dynamics.

TABLE 1: $J^*(z,u)$ FOR EXAMPLE 1

Methods	J
Mamehrashi and Yousefi (2016)	
$m=1, n=4$	0.081044
$m=2, n=4$	0.028790
$m=2, n=7$	0.016484
$m=3, n=7$	0.013027
Hosseini (2022)	
$m=3, n=2$	0.029328
$m=3, n=3$	0.016930
$m=3, n=5$	0.0720
$m=4, n=5$	0.0665
Presented Method	
$M=2, N=2$	0.0104436
$M=2, N=4$	0.00359086
$M=3, N=7$	0.00292532
$M=4, N=4$	0.00190655

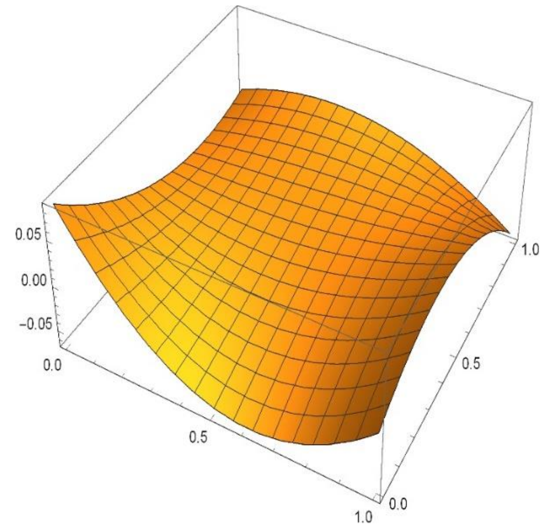

Figure 1: State function z(x,t) with N=2,M=2.

Example 2. Consider the following OCPPC:

$$\min J=\frac{1}{2}\int_0^1\int_0^1 x^2\left[z^2(x,t)+u^2(x,t)\right], dx, dt.$$

The dynamic constraint of the system is given by

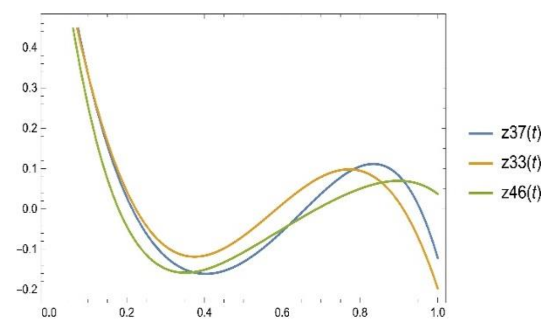

Figure 2: Control function $u(x,t)$ with $N=2, M=2$.

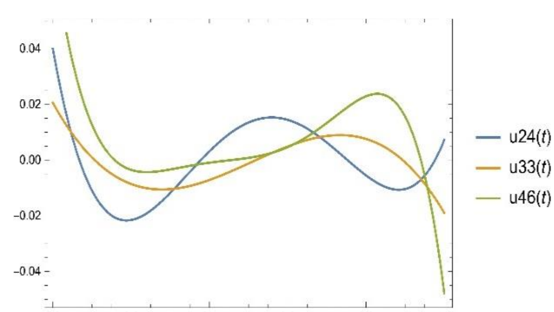

Figure 3: State function $z(t)$ with varying M and N.

$$\frac{\partial z(x,t)}{\partial t}=\frac{\partial^2 z(x,t)}{\partial x^2}+\frac{2}{x}\frac{\partial z(x,t)}{\partial x}+u(x,t), \qquad (19.4)$$

$z(x,0)=\sin(2\pi x); \quad 0<x<1;$

and the boundary condition

$z(1,t) = 0; \quad t > 0.$

Numericall results obtained by applying the proposed method for $J^*(z,u)$ are reported in the table 2. This table includes the computed values of $J(z,u)$ reported by Mamehrashi et al. [6] and Mohammadi et al. [7], as well as the results calculated in this study. As observed, the values of $J^*(z,u)$ obtained in this study demonstrate superior performance compared to those reported. For example, Table 2, in iterations $M = 2, N = 6$, the cost function value obtained by our method is 1.14694×10^{-9}, while for the method proposed in [6] is 0.018283. This shows that our approach yields better results in comparable iterations. Furthermore, when examining the cost function values overall, our method provides superior results in every iteration, outperforming the methods proposed in [6] and [7]. Figures 5 and 6 show the calculated state function $z(x,t)$ and control function $u(x,t)$ for the spectral approximation case with $N = 3$ and $M = 3$. In Figure 5, the surface plot of $z(x,t)$ demonstrates a smooth and continuous structure characterized by slight fluctuations on the order of 10^{-27}. The curvature of the surface indicates the nature of the dynamic response under the given approximation parameters. The nearly flat profile with minimal deviation suggests a system with negligible dynamics under the imposed control or a highly accurate balancing of the state trajectory within the constraints of the spectral method. In contrast, Figure 6 presents the control function $u(z,t)$, which exhibits more pronounced variations. Such

TABLE 2: $J^*(z,u)$ FOR EXAMPLE 2

Methods	J
Mamehrashi and Yousefi (2016)	
$m = 2, n = 4$	0.028790
$m = 2, n = 6$	0.018283
$m = 3, n = 7$	0.013027
$m = 3, n = 9$	0.010405
Mohammadi and Hassani (2018)	
$m_1 = m_2 = n_1 = n_2 = 2$	6.29×10^{-8}
$m_1 = 2, m_2 = 3, n_1 = n_2 = 3$	2.91×10^{-9}
$m_1 = m_2 = n_1 = n_2 = 3$	6.43×10^{-10}
$m_1 = m_2 = n_1 = n_2 = 7$	1.57×10^{-12}
Presented Method	
$M = 2, N = 6$	1.14694×10^{-9}
$M = 2, N = 7$	3.80595×10^{-15}
$M = 3, N = 5$	2.77562×10^{-14}
$M = 3, N = 7$	1.68474×10^{-18}

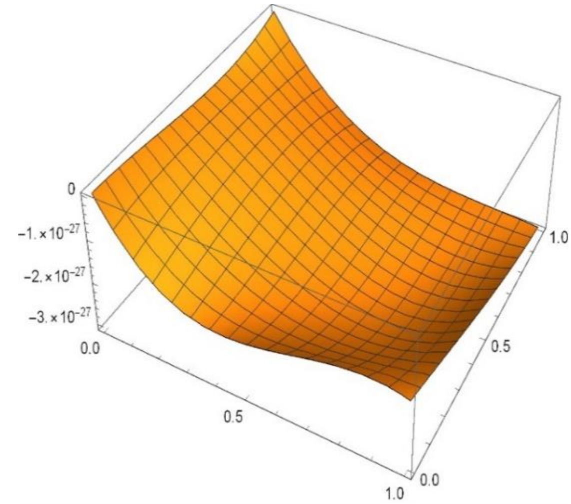

Figure 4: Control function $u(t)$ with varying M and N.

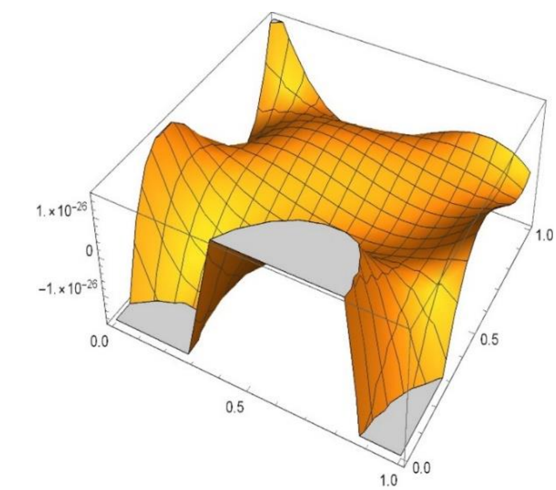

Figure 5: State function $x(t)$ with $N = 3, M = 3$.

behavior is indicative of low-order spectral approximations attempting to capture sharp control transitions or discontinuities in optimal solutions.

VI. Conclusion

This paper presented a novel approach to solving a class of parabolic optimal control problems. The proposed method utilizes a spectral collocation technique, underpinned by Laguerre polynomials, to facilitate the solution process. The proposed method involves expanding the state and control variables in terms of Laguerre basis functions. This transformation of the original optimal control problem into a finite-dimensional nonlinear system can be efficiently solved. The convergence of the proposed scheme was mathematically established, ensuring its theoretical validity. The efficacy of the method was confirmed through a series of numerical experiments, which demonstrated its accuracy, stability, and efficiency. The findings indicated that, even at low approximation orders, the method produces smooth and physically meaningful solutions for both the state and control functions. Furthermore, a comparative analysis with previously published methods in [4] and [6] confirmed the superiority of the proposed approach. As demonstrated by the numerical results, the cost functional value obtained with iteration parameters $M = 3, N = 7$ was significantly lower than those reported in earlier works. The behavior of the state and control functions within their respective domains has been shown to exhibit improved accuracy and convergence with increasing approximation order. The findings indicate the robustness of the proposed technique and underscore its potential for broader applications, including future extensions to multidimensional or fractional-order optimal control problems.

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