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An Analytical Approach to Min-Time Control of Discrete-Time Linear Systems with Constrained Input

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ABSTRACT

Employing discrete-time techniques, the min-time control of continuous-time dynamical systems is mainly studied through an analytical framework. To this aim, the exact discrete-time model of the linear time-invariant systems is specified through a zero-order hold. The optimal solution could be directly determined from some necessary conditions. However, the structure of the optimum control sequences is derived by utilizing the well-known Pontryagin principle. Employing the state transition matrix, the states of the control system are computed at the switching times. The switching times of the control signal would be found from a set of nonlinear algebraic equations. Accordingly, the transformation of the system's states, from a known initial point to a specific value, would be accomplished in the minimum possible time. Applying the proposed scheme, the exact (integer) values of the switching times and the final time are numerically determined from the solution of an algebraic equation. Several discrete-time and continuous-time examples are discussed and simulated to show the feasibility and effectiveness of the suggested procedure in the dynamical systems. The simulation results confirm the method's advantages over the existing ones.

NOMENCLATURE

$x(t)$	The states of continuous-time system	$A_c,\ B_c$	The matrices of continuous-time system
$u(t)$	The inputs of continuous-time system	$A,\ B$	The matrices of discrete -time system
x_k	The states of discrete-time system	m	Number of control inputs
u_k	The inputs of discrete-time system	n	Number of system's states
T_s	Sampling-time	$\|.\|$	Two-norm of a matrix
N	Number of samples	$\det(.)$	Determinant

I. Introduction

In the optimal control, satisfying some constraints on the inputs and the outputs, the profile of the system's input is reshaped to minimize a known cost function. Deriving necessary and sufficient optimality conditions, the optimal control can be formulated in the dynamical descriptions. Most times, there may not be a systematic guideline to the solution of the optimal control. Hence the approximated techniques have been interested in the control applications.

The control efforts may be not optimally found in many applications. Moreover, the optimal control does not have a unique solution. As an example, in the linear time-invariant (LTI) model without input limitation, the min-time control has not an optimum solution [1]. Considering quadratic cost functions, the optimal control of the unconstrained LTI systems has a unique solution. Due to the emergence of a two-point boundary value problem, determining the optimum solution exactly may have some complexities via the analytical ways [1]. Because of such difficulty, it may be of interest to focus on suboptimal or near-to-optimal control policies [2-5]. Numerical methods like the wavelet tool and iterative mechanisms may be utilized to determine the approximated solution [6, 7]. Using semi-definite programming, an optimal control is addressed for piecewise affine systems [8].

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In the min-time problems, the input shape is selected such that the goal would be satisfied in the minimum time. Usually, the bang-bang compensator is reduced to a min-time control. Compared to the other forms like minimum fuel or energy, the solution of the min-time has extra obstacles. The min-time control has been applied in the aerospace vehicles [9]. The optimal guidance issues may be treated as a control system [10-12]. Hence, achieving the exact solutions would be helpful in such cases. The phase plane is a graphical tool for the min-time control in the second-order systems. Although the phase plane is efficient, it cannot be generalized to the high-order models [1]. In the constrained models, a numerical framework is developed for finding the solution of the optimal control problems [13].

There have been some attempts to the min-time issues. As an example, the min-time is studied in controlling a stepping motor [14]. The min-time control is robustly assessed in constrained discrete-time descriptions [15]. Employing a non-smooth optimization, the min-time problem is discussed in discrete-time linear systems [16]. The algebraic solution is examined in min-time velocity planning [17]. The min-time control is optimally found in the LTI model with real poles [18]. The fast solution of the min-time problem is computed in low-thrust transfer with eclipses [19]. Based on lattice theory, the min-time is solved in speed planning [20]. The solution of the HJB min-time is revealed via multilevel techniques [21]. Insuring dead-beat property, an optimal control law is designed in continuous-time dynamical systems [22]. The min-time control of Boolean networks is performed via a fast graphical approach [23]. Concerning smooth transition in communication topology, the minimum time formation control is studied in an underwater vehicle [24]. Based on Pontryagin principle and penalty functions, the predictive control is formulated in continuous-time nonlinear systems [25].

The min-time problem with a limited input signal may not have a solution due to the imposed nonlinearity. However, the optimum control effort can be explicitly obtained in the feasible cases. Recently, relying on a state transition matrix, the min-time control is formulated and solved in the constrained continuous-time systems [26]. Thinking of such an idea, the control input would contain some additive pulses at the calculated time-instants. So, the control aims may disappear due to the sensitivity to the inaccurate and approximated time-instants. Such a drawback can be significantly handled in the discrete-time form. Hence, this point motivates the author to study the min-time property in the LTI plant under input limitations and sampled-data control. Thus the minimum time and the switching instants will be computed by the addressed algebraic equation. As a result, the optimal control signal can be explicitly determined in typical LTI systems. Compared to the previous studies, finding some integer numbers, corresponding to the switching instants, would be the main advantage of the current work. Therefore, the sensitivity of the overall performance is effectively diminished via the discrete-time models. To implement such an idea, the optimal inputs are first computed from an algebraic equation. Then the corresponding differential or difference equations are solved via a recursive scheme.

The structure of the paper is divided as follows: In Section 2, the well-known min-time control is formulated in the discrete-time linear systems with a constrained input. To compute the optimal control signal, the key results of the paper are presented in Section 3. To show the method's feasibility, some examples are analytically reported and simulated in Section 4. The concluding points are briefly provided in the last section.

II. Problem statement

Consider the following constrained LTI systems:

$$\dot{x} = A_c x + B_c u, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $u(t) \in \mathbb{R}^m$ is the control effort. In (1), the input signals are constrained by $|u_i(t)| \leq 1$, $i = 1,2,\dots,m$, while the matrices $A_c \in \mathbb{R}^{n\times n}$ and $B_c \in \mathbb{R}^{n\times m}$ are some given terms. It is desired to find an input $u(t)$ such that the states of the dynamic (1) move from the initial point x_0 to the final value $x(t_f)$ in the minimum possible time. So the cost function that is to be optimized is $J = \int_0^{t_f} 1\mathrm{d}t$.

It is established that the control signal involves some square pulses at some time-instants [26]. Notwithstanding, the switching times should be calculated with special precaution. To handle such a complexity, (1) is discretized under sampling-time T_s. Imagining a zero-order hold (ZOH) in the system's input, the equivalent discrete-time model can be found as

$$x_{k+1} = Ax_k + Bu_k, \tag{2}$$

where $A = e^{A_c T_s}$ and $B = \int_0^{T_s} e^{A_c \tau} B_c \mathrm{d}\tau$. Moreover, the cost function would be rewritten as $J = T_s \sum_{k=0}^{N-1} 1$. Concerning the control limitation ($u_k = [u_{1k} \quad u_{2k} \quad \dots \quad u_{mk}]^T$, $|u_{ik}| \leq 1$, $i = 1,2,\dots,m$) and $t_f = NT_s$, N input control sequences will be determined to transform the system's states from x_0 to x_N in the minimum time.

Assumption 1. To guarantee the control aim in the minimum time, the following condition holds:

$$\|x_N\| \leq \|A\|^N \|x_0\| + \|B\| \frac{\|A\|^N - 1}{\|A\| - 1}. \tag{3}$$

The right hand side of (3) with respect to N would be increasing if $\mathrm{Ln}(\|A\|)\left(\|x_0\| + \frac{\|B\|}{\|A\|-1}\right) > 0$. Based on the discrete-time state transition matrix, to fulfill the control aims, an analytical framework is developed in the next section.

III. Main results

The control aims would be fulfilled by finding some appropriate control signals u_k. To this, the main results of this paper is presented as follows:

Theorem 1. In the LTI system (2) with Assumption 1, for some sequences u_k, $k = 0,1,\dots,n-1$ with $|u_{ik}| = 1$, $i = 1,2,\dots,m$, if there exists integer numbers k_i, $i = 1,\dots,n$, such that the following identity has a solution:

$$\begin{aligned} &x_N + (A-I)^{-1}Bu_{n-1} \\ &= A^{k_n}(x_0 + (A-I)^{-1}Bu_0) \\ &+ \sum_{i=1}^{n-1} A^{k_n-k_i}(A-I)^{-1}B(u_i - u_{i-1}), \end{aligned} \tag{4}$$

where $k_n = N$ and $x_{k_n} = x_N$, then, by applying the control sequence:

$$u_k = \begin{cases} u_0, & 0 \le k < k_1 \\ u_1, & k_1 \le k < k_2 \\ \vdots & \\ u_{n-1}, & k_{n-1} \le k \le k_n \end{cases} \tag{5}$$

the system's states move from x_0 to x_N in the minimum possible time.

Proof. To solve the optimization issue, as reported in [1], the Hamiltonian function is constructed as

$$H = 1 + \lambda_{k+1}^T(Ax_k + Bu_k). \tag{6}$$

Hence the stationary condition would be reduced to

$$\frac{\partial H}{\partial u} = B^T\lambda_{k+1} = 0. \tag{7}$$

As a consequence, it fails to find the optimum inputs. The Pontryagin principle is utilized as follows [27]:

$$H(x_k^*, u_k^*, \lambda_{k+1}^*) \le H(x_k^*, u_k, \lambda_{k+1}^*). \tag{8}$$

Then the following condition is obtained:

$$\begin{aligned} &\lambda_{k+1}^{*T}Bu_k^* \le \lambda_{k+1}^{*T}Bu_k, \ |u_{ik}| \le 1, \\ &i = 1,2,\dots,m. \end{aligned} \tag{9}$$

Accordingly, the optimal control sequences are found as $u_k^* = -\text{sign}\left(\lambda_{k+1}^{*T}B\right)$. Therefore, the control signals are kept within the boundaries of the limitations (i,e,. $u_{ik}^* = \pm 1$, $i = 1,2,\dots,m$). Drawing the phase diagram of the open-loop system corresponding to $u_{ik}^* = +1$ and $u_{ik}^* = -1$, it is shown that there are at most $n-1$ switching instants (namely, $k_1, k_2, \dots, k_{n-1}$) in the control signal [18]. Resultantly, the control sequences are interpreted as the following:

$$u_k = \begin{cases} u_0, & 0 \le k < k_1 \\ u_1, & k_1 \le k < k_2 \\ \vdots & \\ u_{n-1}, & k_{n-1} \le k \le k_n \end{cases} \tag{10}$$

The state values at the mentioned instants are calculated as follows:

$$\begin{cases} x_{k_1} = A^{k_1}x_0 + (A^{k_1-1}B + \cdots + B)u_0 \\ x_{k_2} = A^{k_2-k_1}x_{k_1} + (A^{k_2-k_1-1}B + \cdots + B)u_1 \\ \vdots \\ x_{k_n} = A^{k_n-k_{n-1}}x_{k_{n-1}} \\ \quad + (A^{k_n-k_{n-1}-1}B + \cdots + B)u_{n-1} \end{cases} \tag{11}$$

where $k_n = N$ and $x_{k_n} = x_N$.

The explicit response of the discrete-time system (2) is

$$x_k = A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1}Bu_i. \tag{12}$$

Utilizing the triangular inequality, the two-norm of x_k may be written as:

$$\|x_k\| \le \|A\|^k\|x_0\| + \sum_{i=0}^{k-1}\|A\|^{k-i-1}\|B\|. \tag{13}$$

So the last condition would be directly obtained from (12). In the condition which $\det(A-I) \ne 0$ holds, we have:

$$\begin{aligned} &I + A + A^2 + \cdots + A^{q-1} = (A^q - I)(A-I)^{-1} \\ &= (A-I)^{-1}(A^q - I). \end{aligned} \tag{14}$$

The optimal control may have not a solution. For example, there is not a feasible solution in the following conditions:

$$\|x_N\| > \|A\|^N\|x_0\| + \|B\|\frac{\|A\|^N - 1}{\|A\| - 1}. \tag{15}$$

The right side of (15) with respect to N would be increasing if $\|A\| > 1$ holds. Moreover, guaranteeing $\|A\| < 1$, such the sequence could be increasing if $\frac{\|B\|}{1-\|A\|} < \|x_N\| - \|x_0\|$. Using the property $\|A\|^N < 1$, the condition $\|x_0\| + \|B\|\frac{1}{1-\|A\|} < \|x_N\|$ leads to (15).

The identities (11) are characterized as:

$$\begin{cases} x_{k_1} = A^{k_1}x_0 + (A^{k_1} - I)(A-I)^{-1}Bu_0 \\ x_{k_2} = A^{k_2-k_1}x_{k_1} + (A^{k_2-k_1} - I)(A-I)^{-1}Bu_1 \\ x_{k_3} = A^{k_3-k_2}x_{k_2} + (A^{k_3-k_2} - I)(A-I)^{-1}Bu_2 \\ \vdots \\ x_{k_n} = A^{k_n-k_{n-1}}x_{k_{n-1}} \\ \quad + (A^{k_n-k_{n-1}} - I)(A-I)^{-1}Bu_{n-1} \end{cases} \tag{16}$$

The identity (11) may be modified in a compact form. As an example, for the second-order case (i.e., $n = 2$), we have:

$$\begin{aligned} &x_{k_2} = A^{k_2}x_0 + (A^{k_2} - A^{k_2-k_1})(A-I)^{-1}Bu_0 \\ &+ (A^{k_2-k_1} - I)(A-I)^{-1}Bu_1. \end{aligned} \tag{17}$$

Thus

$$\begin{aligned} &x_{k_2} + (A-I)^{-1}Bu_1 \\ &= A^{k_2}(x_0 + (A-I)^{-1}Bu_0) \\ &+ A^{k_2-k_1}(A-I)^{-1}B(u_1 - u_0). \end{aligned} \tag{18}$$

Similarly, for the third order model (i.e., $n = 3$), we have:

$$\begin{aligned} &x_{k_3} = A^{k_3}x_0 + (A^{k_3} - A^{k_3-k_1})(A-I)^{-1}Bu_0 \\ &+ (A^{k_3-k_1} - A^{k_3-k_2})(A-I)^{-1}Bu_1 \\ &+ (A^{k_3-k_2} - I)(A-I)^{-1}Bu_2. \end{aligned} \tag{19}$$

Then

$$\begin{aligned} &x_{k_3} + (A-I)^{-1}Bu_2 \\ &= A^{k_3}(x_0 + (A-I)^{-1}Bu_0) \\ &+ A^{k_3-k_1}(A-I)^{-1}B(u_1 - u_0) \\ &+ A^{k_3-k_2}(A-I)^{-1}B(u_2 - u_1). \end{aligned} \tag{20}$$

Therefore, for a typical order n, the optimality condition is generalized as:

$$\begin{aligned} &x_N + (A-I)^{-1}Bu_{n-1} \\ &= A^{k_n}(x_0 + (A-I)^{-1}Bu_0) \\ &+ \sum_{i=1}^{n-1} A^{k_n-k_i}(A-I)^{-1}B(u_i - u_{i-1}). \end{aligned} \tag{21}$$

It completes the proof. □

Remark 1. The addressed strategy can be divided to $2^{m\times n}$ cases. In each one, it leads to a set of n variables and n equations. Thus the min-time solution would be exactly found for a typical description. Moreover, such a process is performed in an off-line way. Accordingly, the complexity due

to the execution time may be handled via powerful software packages along with upgrading the hardware.

Remark 2. Although the above closed-forms are corresponding to $\det(A-I)\neq 0$, employing (11), similar conclusions can be obtained for the discrete-time systems with $\det(A-I)=0$.

Remark 3. Mathematically, the transition time among the switching signal happens in zero seconds. However, jumping from $u_{ik}=-1$ to $u_{ik}=+1$ may be challenging points in practice. Additionally, by smoothing such a sharpness, the minimum time property would disappear in the system's response.

IV. Numerical examples

In this section, to validate the applicability of the suggested procedure, several min-time examples are studied in discrete-time and continuous-time models. All simulations are performed via Matlab R2016b.

Example 1. Consider the discrete-time model (2) with the following data:

$$A=\begin{bmatrix}0.7 & 0.3\\ 0.1 & 0.5\end{bmatrix},\quad B=\begin{bmatrix}0.1\\ 0.5\end{bmatrix}.$$

It is of interest to move the system's state from $x_0=[1\quad -1]^T$ to $x_N=[3\quad 1]^T$ in a minimum time. To find the optimal control, the state transition matrix is calculated as the following:

$$A^k=\mathcal{Z}^{-1}\{(zI-A)^{-1}z\}=\begin{bmatrix}\frac{3}{4}(0.8)^k+\frac{1}{4}(0.4)^k & \frac{3}{4}(0.8)^k-\frac{3}{4}(0.4)^k\\ \frac{1}{4}(0.8)^k-\frac{1}{4}(0.4)^k & \frac{1}{4}(0.8)^k+\frac{3}{4}(0.4)^k\end{bmatrix}. \quad (22)$$

The optimum solution would be computed from the following equation:

$$x_N+(A-I)^{-1}Bu_1=A^{k_2}(x_0+(A-I)^{-1}Bu_0)+A^{k_2-k_1}(A-I)^{-1}B(u_1-u_0). \quad (23)$$

Then

$$\begin{bmatrix}9\\3\end{bmatrix}-\begin{bmatrix}5\\4\end{bmatrix}u_1=A^{k_2}\left(\begin{bmatrix}3\\-3\end{bmatrix}-\begin{bmatrix}5\\4\end{bmatrix}u_0\right)-A^{k_2-k_1}\begin{bmatrix}5\\4\end{bmatrix}(u_1-u_0). \quad (24)$$

Subsequently, the identity (24) is investigated in the following cases:

A) When $u_0=+1$ and $u_1=+1$, then (24) may lead to $36(0.8)^{k_2}=-12$ and $19(0.4)^{k_2}=7$.

B) If $u_0=+1$ and $u_1=-1$, then (24) would be reduced to

$$\begin{cases}2(0.8)^{k_2-k_1}-(0.8)^{k_2}=\frac{7}{3}\\ 19(0.4)^{k_2}-14(0.4)^{k_2-k_1}=-7\end{cases}. \quad (25)$$

C) Substituting $u_0=-1$ and $u_1=+1$, (24) is simplified as

$$\begin{cases}(0.8)^{k_2}-2(0.8)^{k_2-k_1}=\frac{1}{3}\\ 5(0.4)^{k_2}+14(0.4)^{k_2-k_1}=7\end{cases}. \quad (26)$$

D) If $u_0=-1$ and $u_1=-1$, then (24) would lead to $36(0.8)^{k_2}=84$ and $5(0.4)^{k_2}=-7$.

Accordingly, it is clear that there is not a solution in all the cases. This problem may be checked by concerning the allowable upper bound as the following:

$$\xi_N=\|A\|^N\|x_0\|+\|B\|\frac{\|A\|^N-1}{\|A\|-1},\ N\geq 1. \quad (27)$$

In this example, we have:

$\|A\|=0.8319,\ \|B\|=0.5099,\ \|x_0\|=1.4142.$

Thus (27) is determined as

$$\xi_N=3.0333-1.6191(0.8319)^N,\ N\geq 1. \quad (28)$$

Checking $\mathrm{Ln}(\|A\|)\left(\|x_0\|+\frac{\|B\|}{\|A\|-1}\right)=0.298>0$, it is clear that ξ_N is an increasing sequence ($\xi_N<3.0333$). The mentioned condition is obtained via differentiation of $\xi_N=\|A\|^N\|x_0\|+\|B\|\frac{\|A\|^N-1}{\|A\|-1}$ with respect to N. In Example 1, $\|x_N\|=3.1623>\xi_N$. So, Assumption 1 does not hold and there is not a solution to the min-time control of Example 1.

Example 2. Consider the min-time control of the following discrete-time model:

$$A=\begin{bmatrix}0.7 & 0.3\\ 0.1 & 0.5\end{bmatrix},\quad B=\begin{bmatrix}1\\2\end{bmatrix},\ x_0=\begin{bmatrix}1\\-1\end{bmatrix}\text{ and }x_N=\begin{bmatrix}-7\\-5\end{bmatrix}.$$

In Example 2, $\|A\|=0.8319$, $\|B\|=2.2361$, and $\|x_0\|=1.4142$. Using Assumption 1, (3) is reduced to $(0.8319)^N\leq 0.3953$. So, there exists the integer $N\geq 6$ such that Assumption 1 holds. Similar to Example 1, the state transition matrix is

$$A^k=\begin{bmatrix}\frac{3}{4}(0.8)^k+\frac{1}{4}(0.4)^k & \frac{3}{4}(0.8)^k-\frac{3}{4}(0.4)^k\\ \frac{1}{4}(0.8)^k-\frac{1}{4}(0.4)^k & \frac{1}{4}(0.8)^k+\frac{3}{4}(0.4)^k\end{bmatrix}.$$

The optimal control is derived via the following algebraic equation:

$$x_N+(A-I)^{-1}Bu_1=A^{k_2}(x_0+(A-I)^{-1}Bu_0)+A^{k_2-k_1}(A-I)^{-1}B(u_1-u_0).$$

Then

$$\begin{bmatrix}-21\\-15\end{bmatrix}-25\begin{bmatrix}1.1\\0.7\end{bmatrix}u_1=A^{k_2}\left(\begin{bmatrix}3\\-3\end{bmatrix}-25\begin{bmatrix}1.1\\0.7\end{bmatrix}u_0\right)-A^{k_2-k_1}25\begin{bmatrix}1.1\\0.7\end{bmatrix}(u_1-u_0). \quad (29)$$

To quantify the optimum solution, the following cases are taken into account:

A) If $u_0=+1$ and $u_1=+1$, then (29) would lead to $(0.8)^{k_2}=1.8$ and $37(0.4)^{k_2}=49$.

B) When $u_0=+1$ and $u_1=-1$, then (29) may be reduced to

$$\begin{bmatrix}26\\10\end{bmatrix}=\begin{bmatrix}-135(0.8)^{k_2}+37(0.4)^{k_2}+270(0.8)^{k_2-k_1}-50(0.4)^{k_2-k_1}\\ -45(0.8)^{k_2}-37(0.4)^{k_2}+90(0.8)^{k_2-k_1}+50(0.4)^{k_2-k_1}\end{bmatrix}$$

It may be modified as:

$$\begin{cases}2(0.8)^{k_2-k_1}-(0.8)^{k_2}=0.2\\ 50(0.4)^{k_2-k_1}-37(0.4)^{k_2}=1\end{cases}. \quad (30)$$

C) If $u_0=-1$ and $u_1=+1$, then (29) could be simplified as:

$$-4\begin{bmatrix}48.5\\32.5\end{bmatrix}$$
$$=\begin{bmatrix}135(0.8)^{k_2}-13(0.4)^{k_2}-270(0.8)^{k_2-k_1}+50(0.4)^{k_2-k_1}\\45(0.8)^{k_2}+13(0.4)^{k_2}-90(0.8)^{k_2-k_1}-50(0.4)^{k_2-k_1}\end{bmatrix}$$

It equals to

$$\begin{cases}2(0.8)^{k_2-k_1}-(0.8)^{k_2}=1.8\\50(0.4)^{k_2-k_1}-13(0.4)^{k_2}=49\end{cases}. \quad (31)$$

D) Substituting $u_0=-1$ and $u_1=-1$, (29) leads to $(0.8)^{k_2}=0.2$ and $13(0.4)^{k}=1$.

Consequently, no solution is anticipated in Cases A, C and D for (29). There is also a solution for Case B. But it is not an integer. However, taking $k_1=8$ and $k_2=18$, the final value $x_N=[-6.9538\quad -5.0951]^T$ would be touched in Example 2. The induced errors are directly related to the integer values of k_1 and k_2. Although it is not an exact solution, it is sufficiently near to the desired final point. As shown in Fig 1, the optimum control sequences are found as

$$u_k=\begin{cases}+1,\ 0\le k<8\\-1,\ 8\le k<18\end{cases} \quad (32)$$

Moreover, the states of Example 2 are illustrated in Fig 1.

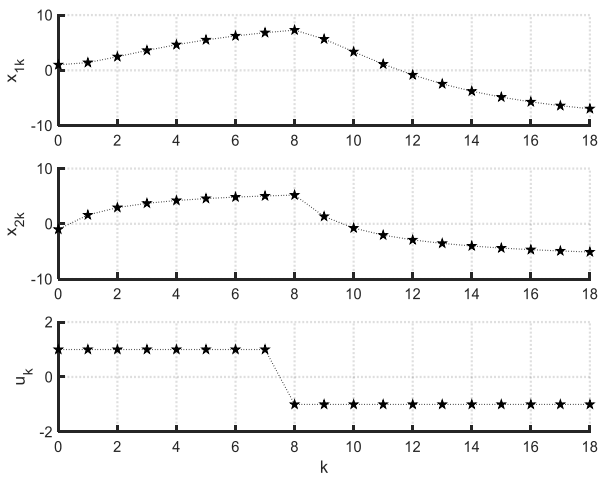


Fig 1. The states and control input of Example 2

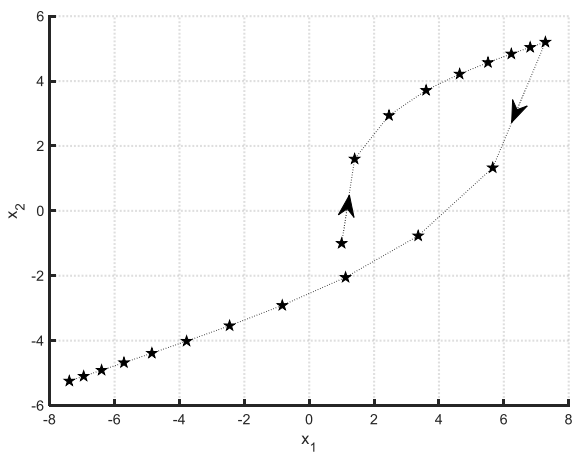


Fig 2. The state's trajectory in Example 2

In the phase plane, the corresponding trajectory is plotted as seen in Fig 2. Accordingly, the control aims are effectively verified via the min-time control signal (32).

Example 3. Consider the following second-order continuous-time dynamic:

$$\ddot{y}=u, \quad (33)$$

where y is the position and u is the acceleration. The acceleration is limited by $|u(t)|\le 1$. It is desired to move the system's states from $y(0)=1$, $\dot{y}(0)=-1$ to $y(t_f)=2$, $\dot{y}(t_f)=0$ in the possible minimum time. The state-space representation is

$$\dot{x}=\begin{bmatrix}0&1\\0&0\end{bmatrix}x+\begin{bmatrix}0\\1\end{bmatrix}u. \quad (34)$$

Including the block ZOH, the equivalent discrete-time description is

$$x_{k+1}=\begin{bmatrix}1&T_s\\0&1\end{bmatrix}x_k+\begin{bmatrix}\frac{1}{2}T_s^2\\T_s\end{bmatrix}u_k. \quad (35)$$

Taking the sampling-time $T_s=1$, the discrete-time model is found as:

$$A=\begin{bmatrix}1&1\\0&1\end{bmatrix},\ B=\begin{bmatrix}\frac{1}{2}\\1\end{bmatrix} \quad (36)$$

The condition (3) is simplified as $1.1817\le(1.6180)^N$. So, Assumption 1 can hold for integer $N\ge 1$ in Example 3. To move the states from $x_0=[1\quad -1]^T$ to $x_N=[2\quad 0]^T$ in a minimum time, the optimality condition is written as the following:

$$\begin{cases}x_{k_1}=A^{k_1}x_0+(A^{k_1-1}+\cdots+I)Bu_0\\x_N=A^{k_2-k_1}x_{k_1}+(A^{k_2-k_1-1}+\cdots+I)Bu_1\end{cases} \quad (37)$$

The singular condition $\det(A-I)=0$ takes place in this example. The condition (37) can be written as:

$$x_N=A^{k_2}x_0+A^{k_2-k_1}(A^{k_1-1}+\cdots+I)Bu_0 \\ +(A^{k_2-k_1-1}+\cdots+I)Bu_1. \quad (38)$$

The state transition matrix is disclosed as

$$A^k=\begin{bmatrix}1&k\\0&1\end{bmatrix}. \quad (39)$$

The identity (38) is reduced to

$$\begin{bmatrix}2\\0\end{bmatrix}=\begin{bmatrix}1&k_2\\0&1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} \\ +\begin{bmatrix}1&k_2-k_1\\0&1\end{bmatrix}\left(\begin{bmatrix}1&k_1-1\\0&1\end{bmatrix}+\cdots+I\right)\begin{bmatrix}\frac{1}{2}\\1\end{bmatrix}u_0 \\ +\left(\begin{bmatrix}1&k_2-k_1-1\\0&1\end{bmatrix}+\cdots+I\right)\begin{bmatrix}\frac{1}{2}\\1\end{bmatrix}u_1. \quad (40)$$

It may be simplified as:

$$\begin{bmatrix}2\\0\end{bmatrix}=\begin{bmatrix}1&k_2\\0&1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} \\ +\begin{bmatrix}1&k_2-k_1\\0&1\end{bmatrix}\begin{bmatrix}k_1&\frac{k_1(k_1-1)}{2}\\0&k_1\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\1\end{bmatrix}u_0 \\ +\begin{bmatrix}k_2-k_1&\frac{(k_2-k_1)(k_2-k_1-1)}{2}\\0&k_2-k_1\end{bmatrix}\begin{bmatrix}\frac{1}{2}\\1\end{bmatrix}u_1. \quad (41)$$

Then

$$\begin{cases}(2k_2-k_1)k_1u_0+(k_2-k_1)^2u_1=2+2k_2\\k_1u_0+(k_2-k_1)u_1=1\end{cases} \quad (42)$$

To determine the optimal control, the following cases are briefly studied:

A) $u_0=+1,\ u_1=+1$

B) $u_0 = +1,\ u_1 = -1$
C) $u_0 = -1,\ u_1 = +1$
D) $u_0 = -1,\ u_1 = -1$

There is no solution for Cases A, C and D. However, $k_1 = 1 + \frac{\sqrt{6}}{2} = 2.2247$ and $k_2 = 1 + \sqrt{6} = 3.4495$ is found for Case B. Unfortunately, it is not an integer number. Nevertheless, choosing $k_1 = 2$ and $k_2 = 3$, the continuous-time control effort is specified as follows:

$$u(t) = \begin{cases} +1, & 0 \le t < 2 \\ -1, & 2 \le t < 3 \end{cases}. \tag{43}$$

Applying the control signal (43), the final states of (33) are touched at $x_N = [1.5 \quad 0]^T$, while it slightly differs from the desired point. Such an error is specifically caused due to the integer values of the switching instants. To show the merit of the suggested idea, the results are compared with a similar continuous-time approach [26]. In both cases, the states and control efforts of the model are depicted in Fig 3. In the phase plane, the trajectory is plotted in Fig 4.

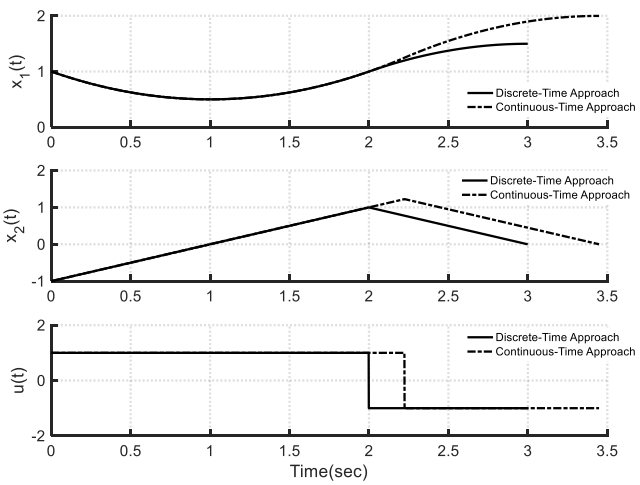


Fig 3. The states and control input in Example 3

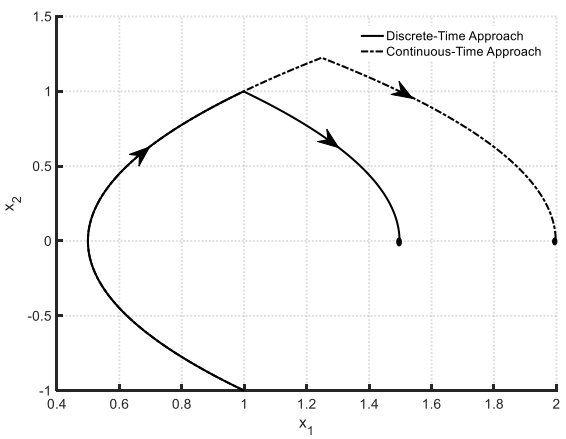


Fig 4. The state's trajectory in Example 3

Compared to a similar idea [26], finding integer switching times would be the main advantage of the suggested control methodology.

Example 4. Consider the min-time control of the following system:

$$\ddot{y} = 2u, \tag{44}$$

with $y(0) = 1$, $\dot{y}(0) = 0$, $y(t_f) = 8$, $\dot{y}(t_f) = -10$ and $|u(t)| \le 1$. The state-space equation would be found as the following:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u. \tag{45}$$

The equivalent discrete model is

$$x_{k+1} = \begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} T_s^2 \\ 2T_s \end{bmatrix} u_k. \tag{46}$$

Taking the sampling period $T_s = 1$, we have:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ x_N = \begin{bmatrix} 8 \\ -10 \end{bmatrix}.$$

The inequality (3) is found as $3.5564 \le (1.6180)^N$. So, Assumption 1 insures for any integer $N > 3$. The optimal control is found from the following conditions:

$$\begin{cases} x_{k_1} = A^{k_1} x_0 + (A^{k_1 - 1} + \cdots + I) B u_0 \\ x_N = A^{k_2 - k_1} x_{k_1} + (A^{k_2 - k_1 - 1} + \cdots + I) B u_1 \end{cases} \tag{47}$$

It is written as:

$$\begin{aligned} x_N &= A^{k_2} x_0 + A^{k_2 - k_1}(A^{k_1 - 1} + \cdots + I) B u_0 \\ &+ (A^{k_2 - k_1 - 1} + \cdots + I) B u_1. \end{aligned} \tag{48}$$

In this case, the state transition matrix is given by

$$A^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$

The identity (48) is arranged as:

$$\begin{aligned} \begin{bmatrix} 8 \\ -10 \end{bmatrix} &= \begin{bmatrix} 1 & k_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &+ \begin{bmatrix} 1 & k_2 - k_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 & \frac{k_1(k_1 - 1)}{2} \\ 0 & k_1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} u_0 \\ &+ \begin{bmatrix} k_2 - k_1 & \frac{(k_2 - k_1 - 1)(k_2 - k_1)}{2} \\ 0 & k_2 - k_1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} u_1. \end{aligned} \tag{49}$$

Then

$$\begin{cases} (2k_1 k_2 - k_1^2) u_0 + (k_2 - k_1)^2 u_1 = 7 \\ 2k_1 u_0 + (2k_2 - 2k_1) u_1 = -10 \end{cases}. \tag{50}$$

To determine the optimal control, (50) is solved in the following cases:

A) $u_0 = +1,\ u_1 = +1$
B) $u_0 = +1,\ u_1 = -1$
C) $u_0 = -1,\ u_1 = +1$
D) $u_0 = -1,\ u_1 = -1$

It is seen that there is no feasible solution for Cases A, C and D. However, $k_1 = 4$ and $k_2 = 13$ is found for Case B. In the continuous-time form, the control signal is depicted as:

$$u(t) = \begin{cases} +1, & 0 \le t < 4 \\ -1, & 4 \le t < 13 \end{cases} \tag{51}$$

The states and input of the model (45) are computed as depicted in Fig 5. Furthermore, the trajectory is plotted in the phase plane as seen in Fig 6.

In Example 4, since the switching instants are integer, the outcomes of the proposed discrete-time idea would be the same as the continuous-time approach [26]. Thus the required

objectives are completely fulfilled via the calculated control effort (51).

In low-order systems, (4) can be solved via the analytic methods as seen in Examples 1-4. However, in complex equations, the nonlinear problem would be solved via numerical iterative methods. Moreover, Matlab's commands like fzero, fsolve, and others can be utilized to find the roots of (4).

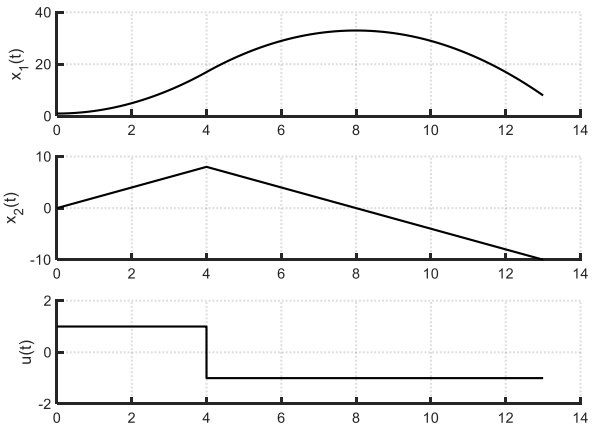


Fig 5. The states and control input in Example 4

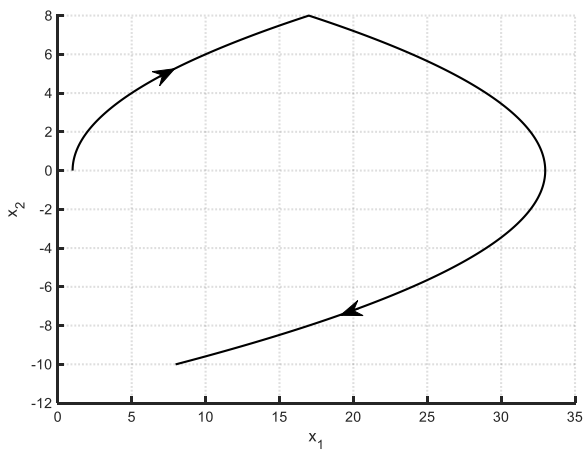


Fig 6. Trajectory in Example 4

Example 5. Consider a discrete-time min-time problem with the following data:

$$A=\begin{bmatrix}0.6 & 0.7 & 0\\ 0 & -0.3 & 0.3\\ -0.2 & 0.2 & 0.4\end{bmatrix},\quad B=\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix},\quad x_0=\begin{bmatrix}2\\ -1\\ 3\end{bmatrix}$$ and $x_N=\begin{bmatrix}4.3\\ 1.4\\ 2.6\end{bmatrix}$.

Employing Assumption 1 with $\|A\|=0.9541$, $\|B\|=1.7321$, and $\|x_0\|=3.7417$, (3) is reduced to $(0.9541)^N\le 0.9566$. So, there exists always integer N for satisfying Assumption 1. In this case, the optimal control would be derived from the following equation:

$$x_N+(A-I)^{-1}Bu_2=A^{k_3}(x_0+(A-I)^{-1}Bu_0)$$
$$+A^{k_3-k_1}(A-I)^{-1}B(u_1-u_0)$$
$$+A^{k_3-k_2}(A-I)^{-1}B(u_2-u_1).$$

Then

$$\begin{bmatrix}4.3\\ 1.4\\ 2.6\end{bmatrix}+\begin{bmatrix}-2.8182 & -1.9091\\ -0.1818 & -1.0909\\ -0.7879 & -1.3939\end{bmatrix}u_2$$
$$=A^{k_3}\left(\begin{bmatrix}2\\ -1\\ 3\end{bmatrix}+\begin{bmatrix}-2.8182 & -1.9091\\ -0.1818 & -1.0909\\ -0.7879 & -1.3939\end{bmatrix}u_0\right)$$
$$+A^{k_3-k_1}\begin{bmatrix}-2.8182 & -1.9091\\ -0.1818 & -1.0909\\ -0.7879 & -1.3939\end{bmatrix}(u_1-u_0)$$
$$+A^{k_3-k_2}\begin{bmatrix}-2.8182 & -1.9091\\ -0.1818 & -1.0909\\ -0.7879 & -1.3939\end{bmatrix}(u_2-u_1).$$

Accordingly, among all the possible cases, a feasible solution of the mentioned equation is numerically found as $k_1=3$, $k_2=8$ and $k_3=12$ which corresponded to $u_0=[-1\quad +1]^T$, $u_1=[+1\quad -1]^T$ and $u_2=[+1\quad +1]^T$. Thus the optimum sequences are obtained as follows:

$$u_{1k}=\begin{cases}-1, & 0\le k<3\\ +1, & 3\le k\le 12\end{cases} \tag{52}$$

and

$$u_{2k}=\begin{cases}+1, & 0\le k<3\\ -1, & 3\le k<8\\ +1, & 8\le k\le 12\end{cases} \tag{53}$$

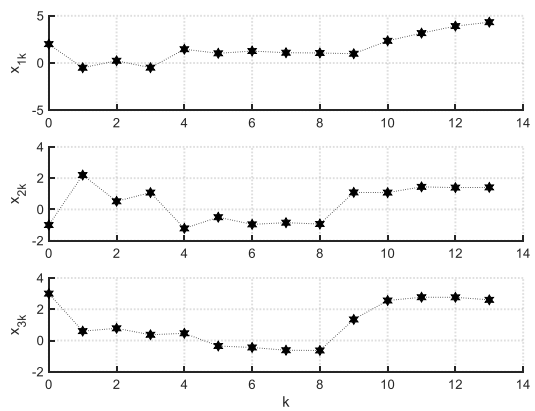


Fig 7. The states of the discrete-time model in Example 5

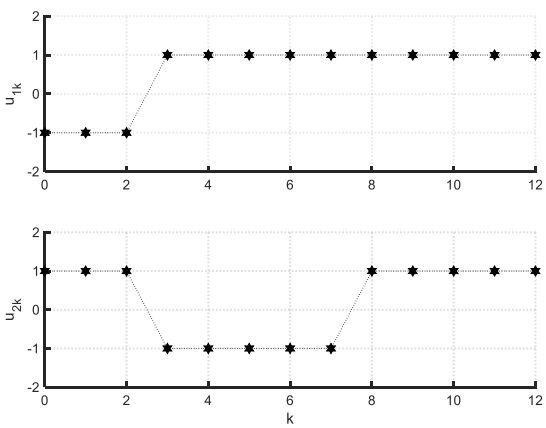


Fig 8. The calculated control inputs in Example 5

Solving the discrete-time model, the states of Example 5 are illustrated in Fig 7. Moreover, the calculated control inputs are

plotted in Fig 8. Consequently, the requirements are fulfilled in Example 5 through the computed control inputs.

V. Conclusion

The movement of the system's states in the minimum time is analytically investigated in the sampled-data control description. Hence, the amplitude of the optimum control signal is firstly derived from the Pontryagin principle. Using the state transition matrix, the system's states are computed at the switching times. Finally, corresponding to the switching times, some integer parameters are exactly determined via the addressed algebraic equation. Moreover, the minimum final time is calculated for the optimal control problem. The effectiveness of the suggested procedure is verified in several discrete-time and continuous-time control examples.

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