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An Operational Matrix Method Based on the Gegenbauer Polynomials for Solving a Class of Fractional Optimal Control Problems

Farzaneh Soufivand [1], Fahimeh Soltanian [2,†], and Kamal Mamehrashi [3]

[1,2,3] Department of Mathematics, Payame Noor University, Tehran, Iran.

ABSTRACT

One of the most important classes of fractional calculus is the fractional optimal control problem (FOCP), which arises in engineering. This study presents a direct and efficient numerical method for solving a class of (FOCPs) in which the fractional derivative is in the Caputo sense and the dynamic system includes the fractional- and integer-order derivatives. For this purpose, we use the operational matrix of fractional Riemann-Liouville integration based on the shifted Gegenbauer polynomials. First, the fractional- and integer-order derivatives in the given problem are approximated based on the shifted Gegenbauer polynomials with unknown coefficients. Then by substituting these approximations and the equation derived from the dynamic constraint into the cost functional, an unconstrained optimization problem is obtained. The main advantage of this approach is that it reduces the FOCP given to an unconstrained optimization problem and using the necessary optimality conditions yields a system of algebraic equations which can be easily solved by Newton's iterative method. In addition, the convergence of the method is proved via several theorems. Finally, some numerical examples are presented to illustrate the validity and applicability of the proposed technique.

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I. INTRODUCTION

Optimal control theory has a wide range of applications in different science and engineering fields [1], such as physics and quantum dynamics [2, 3].

Nowadays, fractional calculus is a very important branch of mathematics and the dynamics of many systems can be described more precisely using fractional order derivatives rather than integer-order derivatives, such as dynamical procedures containing heat conduction and gas diffusion in fractal porous media [4].

Moreover, fractional calculus has numerous real applications in science and engineering fields, such as physics [5], medicine [6], electromagnetism [7], and the dynamic of viscoelastic materials [8]. One of the important topics in fractional calculus is the fractional optimal control problem (FOCP), in which the dynamic constraints as well as the cost functional might entail not only integer-order derivatives but also fractional-order derivatives or integrals [9]. FOCPs can be established with respect to different types of fractional derivatives. However, the most important types of fractional derivatives are the Riemann-Liouville and the Caputo fractional derivatives. Solving FOCPs was first accomplished by Agrawal [10]. Then, in [11], he presented a direct numerical method for solving Riemann-Liouville FOCPs. In the last two decades, orthogonal polynomials have been used for solving different problems [12, 13, 14]. In addition, solving FOCPs with polynomials and

[†]**Corresponding Author**: f_soltanian@pnu.ac.ir
Tel: +98-9131051361, Department of Mathematics, Payame Noor University, Tehran, Iran

operational matrices has received great attention from researchers and thus numerous studies have been conducted in this field. For instance, Legendre polynomials [15, 16], Jacobi polynomials [17, 18], Bernoulli polynomials [19], Bernstein polynomials [20, 21], and Genocchi polynomials [22] have been used for solving different models of FOCPs so far. In [23], by employing an indirect method and the necessary optimality conditions, the FOCP is converted to a two-point boundary value problem, which is solved by the collocation method based on Bernoulli hybrid functions. The Gegenbauer polynomials have many beneficial and important properties and the theory of these polynomial approximations has received considerable attention in recent years [24, 25, 14]. In [26], the Gegenbauer polynomials were employed to solve FOCPs with the Riemann-Liouville fractional derivative. In this paper, we consider the following FOCP in the Caputo sense [20, 21]

$$\min J = \int_0^1 f\big(t, x(t), u(t)\big)dx \tag{1}$$

subject to the dynamic constraint with fractional- and integer-order derivatives

$$Mx'(t) + ND^{\nu}x(t) = a\big(t, x(t)\big) + b(t)u(t), \tag{2}$$

with the boundary conditions

$$x(0) = x_0, \quad x(1) = x_1, \tag{3}$$

where $1 < \nu \leq 2$.

Moreover, we assume that $f\big(t, x(t), u(t)\big), a(t, x(t))$ and $b(t) \neq 0$ are smooth functions, D^{ν} is the Caputo fractional derivative of order ν, and M, and N are constant numbers. The aim of this article is to take the Caputo fractional derivative in the dynamic constraint by constructing the shifted Gegenbauer operational matrix of the Riemann-Liouville fractional integration. To do this and for the optimality of the cost functional, first, we approximate the fractional derivative in the dynamic constraint based on the shifted Gegenbauer polynomials with unknown coefficients. Then, the equation derived from the dynamic constraints is put into the cost functional. In this manner, by applying the necessary optimality conditions, a system of algebraic equations will be obtained that can be easily solved by Newton's iterative method. The remaining part of this paper is organized as follows: In section 2, we introduce some basic definitions of fractional calculus, as well as definitions of shifted Gegenbauer polynomials and their properties, which are required for the subsequent development. In section 3, a new function approximation of FOCPs is constructed. Section 4 presents the shifted Gegenbauer operational matrix of fractional integration. In section 5, we describe the numerical treatment of the proposed method. In section 6, the convergence of the method is given and some theorems are proved. Section 7 reports the numerical results and demonstrates the accuracy of the proposed method through several numerical examples, Moreover, in this section, the numerical results obtained by our method are compared with those found by other methods. Finally, the conclusion is given in section 8.

II. BASIC PRELIMINARIES

In this section, we explain some basic definitions that are required in the following part of the paper.

Definition 1. The Riemann-Liouville fractional integral operator I of order $\nu \geq 0$ on the interval $[0, \infty)$ of a function $g(t)$ is given by [13].

$$I^{\nu}g(t) = \begin{cases} \dfrac{1}{\Gamma(\nu)} \displaystyle\int_0^t (t-y)^{\nu-1} g(y)dy, & \nu > 0, \quad t > 0 \\ g(t), & \nu = 0 \end{cases} \tag{4}$$

The Riemann-Liouville fractional integral has the following properties:

$$I^{\nu}t^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\nu)} t^{k+\nu}, \tag{5}$$

$$I^{\nu}\big(\lambda_1 g_1(t) + \lambda_2 g_2(t)\big) = \lambda_1 I^{\nu} g_1(t) + \lambda_2 I^{\nu} g_2(t) \tag{6}$$

where λ_1, and λ_2 are constants.

Definition 2. The Caputo fractional derivative of order ν of $g(t)$ is given by [13].

$$D^{\nu}g(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-y)^{(n-\nu-1)} \frac{d^n}{dy^n} g(y)dy, \tag{7}$$
$$n-1 < \nu \leq n, n \epsilon N$$

The Caputo fractional derivative D^{ν} has the following properties:

$$D^{\nu}t^n = \begin{cases} \frac{\Gamma(n+1)}{\Gamma(n+1-\nu)} t^{n-\nu}, & n \epsilon N \cup \{0\}, \ n \geq [\nu] \\ 0, & n \epsilon N \cup \{0\}, \ n < [\nu] \end{cases} \tag{8}$$

$$D^{\nu}\big(\lambda_1 g_1(t) + \lambda_2 g_2(t)\big) = \lambda_1 D^{\nu} g_1(t) + \lambda_2 D^{\nu} g_2(t) \tag{9}$$

where λ_1, and λ_2 are constants.

$$D^{\nu}I^{\nu}g(t) = g(t), \quad g(t) \epsilon C^n[0,1] \tag{10}$$

$$I^{\nu}D^{\nu}g(t) = g(t) - \sum_{i=0}^{n-1} g^i(0^+) \frac{t^i}{i!}, \tag{11}$$

$$g(t) \epsilon C^n[0,1], t > 0 \tag{12}$$

We first introduce the Gegenbauer polynomials, and then with a given variable change, we obtain the shifted Gegenbauer polynomials. The Gegenbauer polynomials $C_n^{(\beta)}(t)$ are a set of real-valued polynomials of the degree $n \epsilon Z^+$ associated with the parameter $\beta > -\frac{1}{2}$ in the domain $[-1, 1]$. These polynomials are defined by [14].

$$C_n^{(\beta)}(t) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k \Gamma(n-k+\beta)}{k!\,\Gamma(\beta)(n-2k)!} (2t)^{n-2k} \tag{13}$$

Another way to generate the Gegenbauer polynomials is to use the following recurrence formula:

$$(n+2\beta)C_{n+1}^{(\beta)}(t) = 2(n+\beta)tC_n^{(\beta)}(t) - nC_{n-1}^{(\beta)}(t), \quad (14)$$

Starting with

$$C_0^{(\beta)}(t) = 1, \quad C_1^{(\beta)}(t) = t. \quad (15)$$

The Gegenbauer polynomials are orthogonal with respect to L^2-space on the interval $[-1,1]$. The orthogonality relation is given by:

$$\int_{-1}^{1} C_m^{(\beta)}(t)C_n^{(\beta)}(t)\,\omega_n^{(\beta)}(t)dt = \eta_n^{(\beta)}(t)\delta_{m,n}, \quad (16)$$

where $\omega_n^{\beta}(t)$ and $\eta_n^{\beta}(t)$ are the weight function and the normalizing factor, respectively, and defined by:

$$\omega_n^{(\beta)}(t) = (1-t^2)^{\left(\beta-\frac{1}{2}\right)}, \quad (17)$$

$$\eta_n^{(\beta)}(t) = ||C_n^{(\beta)}(t)||^2 = \frac{2^{1-2\beta}\pi\Gamma(n+2\beta)}{n!(n+\beta)\Gamma^2(\beta)}, \quad (18)$$

and $\delta_{m,n}$ is the Kronecker delta function. In order to use the Gegenbauer polynomials on the interval $[0,1]$, we define the shifted Gegenbauer polynomials by replacing the variable t with $2t-1$, $t\epsilon[0,1]$ [14]. Let us define the shifted Gegenbauer polynomial $C_n^{(\beta)}(2t-1)$ of the degree n by $C_{S,n}^{(\beta)}(t)$, so

$$C_{S,0}^{(\beta)}(t) = 1, \quad C_{S,1}^{(\beta)}(t) = 2t-1. \quad (19)$$

The explicit form of the shifted Gegenbauer polynomial of the degree n is given by [14]:

$$C_{S,n}^{(\beta)}(t) = \frac{\Gamma\left(\beta+\frac{1}{2}\right)}{\Gamma(2\beta)}\sum_{k=0}^{n}\frac{(-1)^{(n-k)}\Gamma(n+k+2\beta)}{(n-k)!\,k!\,\Gamma\left(k+\beta+\frac{1}{2}\right)}(t)^k \quad (20)$$

The orthogonality condition of these polynomials with respect to L^2-space on the interval $[0,1]$ can be obtained by [14]:

$$\int_0^1 C_{S,m}^{(\beta)}(t)C_{S,n}^{(\beta)}(t)\,\vartheta^{\beta}(t)dt = \lambda_n^{(\beta)}(t)\delta_{m,n}, \quad (21)$$

where

$$\vartheta^{\beta}(t) = (t-t^2)^{\left(\beta-\frac{1}{2}\right)}, \quad (22)$$

and

$$\lambda_n^{(\beta)}(t) = \frac{\left(\Gamma\left(\beta+\frac{1}{2}\right)\right)^2\Gamma(n+2\beta)}{\left(\Gamma(2\beta)\right)^2(2n+2\beta)n!} \quad (23)$$

The shifted Gegenbauer polynomials are interrelated with the other polynomials by the following relations:

$$L_n(t) \equiv C_{S,n}^{\left(\frac{1}{2}\right)}(t), \; T_n(t) \equiv C_{S,n}^{(0)}(t), \; U_n^*(t) \equiv C_{S,n}^{(1)}(t), \quad (24)$$

where $L_n(t)$, $T_n(t)$, and $U_n^*(t)$ are the shifted Legendre polynomials, the shifted Chebyshev polynomials of the first kind, and the shifted Chebyshev polynomials of the second kind, respectively.

III. FUNCTION APPROXIMATION

In this section, we will construct an approximate function based on shifted Gegenbauer polynomials with unknown coefficients. Suppose that $H = L^2[0,1]$ is a Hilbert space and define

$$Z = Span\left\{C_{S,0}^{(\beta)}(t), C_{S,1}^{(\beta)}(t), \dots, C_{S,n}^{(\beta)}(t)\right\}. \quad (25)$$

Z is a finite-dimensional and closed subspace of H and therefore Z it is a complete subspace of H. Hence, for every arbitrary element $g(t)$ in H, there exists the unique best approximation out of Z such as $g_n(t)$ that

$$\forall\, z(t)\epsilon Z, \; \left|\left|g(t)-g_n(t)\right|\right|_2 \le \left|\left|g(t)-z(t)\right|\right|_2. \quad (26)$$

We can write, $H = Z \oplus Z^{\perp}$ where $Z^{\perp}$ is the orthogonal complement of Z. So, $g(t) = z(t) + z^{\perp}(t)$ where $z^{\perp}(t) = g(t) - z(t)$. It implies that $g(t) - g_n(t)\epsilon Z^{\perp}$ and therefore

$$\forall\, z(t)\epsilon Z, \; < g(t)-g_n(t), z(t) > = 0, \quad (27)$$

where <: > denotes the inner product. Since $g_n(t)\epsilon Z$, there are unique coefficients $c_0, c_1, \dots, c_n \epsilon R$ such that

$$g(t) \simeq g_n(t) = \sum_{j=0}^{n} c_j\, C_{S,j}^{(\beta)}(t) = C^T\phi(t), \quad (28)$$

where $C = [c_0, c_1, \dots, c_n]^T$,

$$\phi(t) = \left[C_{S,0}^{(\beta)}(t), C_{S,1}^{(\beta)}(t), \dots, C_{S,n}^{(\beta)}(t)\right]^T. \quad (29)$$

Using Eq. (27) and Eq. (28), we have

$$< g(t) - C^T\phi(t), C_{S,i}^{(\beta)}(t) > = 0 \;\; i = 0,1,\dots,n, \quad (30)$$

or equivalently

$$< g(t), \phi(t) > = C^T < \phi(t), \phi(t) > \quad (31)$$

where $< \phi(t), \phi(t) >$ is the following $(n+1)\times(n+1)$ matrix

$$Q = < \phi(t), \phi(t) > = \int_0^1 \phi(t)\phi^T(t)\,\vartheta^{\beta}(t)dt. \quad (32)$$

From Eq. (30) and Eq. (32), we obtain

$$C = Q^{-1} < g(t), \phi(t) >, \quad (33)$$

where

$$c_i = (\lambda_i^{\beta})^{-1}\int_0^1 g(t)\,\vartheta^{\beta}(t)C_{S,i}^{(\beta)}(t)dt\,, i = 0,1,\dots,n. \quad (34)$$

IV. SHIFTED GEGENBAUER OPERATIONAL MATRIX OF FRACTIONAL INTEGRATION

In this section, in order to solve FOCPs (1)-(3) based on shifted Gegenbauer polynomials, we will construct an operational matrix with a given theorem.

Theorem 1: Suppose $\phi(t)$ is the shifted Gegenbauer vector given with Eq. (29) and let $\nu > 0$. Then

$$I^{\nu}\phi(t) \simeq F^{\nu}\phi(t), \quad (35)$$

where F^{ν} is the $(n+1)\times(n+1)$ operational matrix of fractional integration of order ν in the Riemann-Liouville sense, given by:

$$F^{\nu}=\begin{bmatrix} \theta_{0,0,0} & \theta_{0,1,0} & \cdots & \theta_{0,n,0} \\ \sum_{j=0}^{1}\theta_{1,0,j} & \sum_{j=0}^{1}\theta_{1,1,j} & \cdots & \sum_{j=0}^{1}\theta_{1,n,j} \\ \sum_{j=0}^{2}\theta_{2,0,j} & \sum_{j=0}^{2}\theta_{2,1,j} & \cdots & \sum_{j=0}^{2}\theta_{2,n,j} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{j=0}^{n}\theta_{n,0,j} & \sum_{j=0}^{n}\theta_{n,1,j} & \cdots & \sum_{j=0}^{n}\theta_{n,n,j} \end{bmatrix}$$

where $\theta_{i,r,j}$ is as follows:

$$\theta_{i,r,j}=\sum_{j=0}^{i}\frac{(-1)^{i-j}\Gamma\left(\beta+\frac{1}{2}\right)\Gamma(i+j+2(\beta))}{(i-j)!\,\Gamma\left(j+\beta+\frac{1}{2}\right)\Gamma(2\beta)\Gamma(j+\nu+1)}\times$$

$$\sum_{j=0}^{i}\frac{(-1)^{r-k}r!\,(r+\beta)\Gamma^{2}(\beta)\Gamma^{2}\left(\beta+\frac{1}{2}\right)\Gamma\left(\nu+j+k+\beta+\frac{1}{2}\right)}{2^{(1-4\beta)}\pi(r-k)!\,k!\,\Gamma(r+2\beta)\Gamma(2\beta)\Gamma\left(k+\beta+\frac{1}{2}\right)}$$
$$\times\frac{\Gamma(r+k+2\beta)}{\Gamma(\nu+j+k+2\beta+1)} \quad (36)$$

Proof: See [26].

V. NUMERICAL TREATMENT

In this section, we describe our numerical method for solving FOCPs (1)-(3). In Eq. (2), we approximate $D^{\nu}x(t)$ by Gegenbauer polynomials as

$$D^{\nu}x(t)=C^{T}\phi(t), \quad (37)$$

here $\phi(t)$ and C are $(n+1)\times 1$ matrices by Eq. (29). By integrating from (37) and using (35), we obtain

$$I^{\nu}D^{\nu}x(t)=C^{T}F^{\nu}\phi(t), \quad (38)$$

and from (11), we have

$$I^{\nu}D^{\nu}x(t)=x(t)-tx'(0)-x(0), \quad (39)$$

Thus, $x(t)$ can be represented by

$$x(t)=C^{T}F^{\nu}\phi(t)+tx'(0)+x_0, \quad (40)$$

Therefore, the boundary condition $x(1)=x_1$ yields

$$x_1=x(1)=C^{T}F^{\nu}\phi(1)+x'(0)+x_0, \quad (41)$$

or equivalently

$$x'(0)=x_1-x_0-C^{T}F^{\nu}\phi(1), \quad (42)$$

Substituting Eq. (42) into Eq. (40), gives

$$x(t)=C^{T}F^{\nu}\phi(t)+t\left(x_1-x_0-C^{T}F^{\nu}\phi(1)\right)+x_0, \quad (43)$$

and by deriving Eq. (43), we estimate the ordinary derivative of $x(t)$ as

$$x'(t)=C^{T}F^{\nu}D\phi(t)+x_1-x_0-C^{T}F^{\nu}\phi(1), \quad (44)$$

where D is the operational matrix of differentiation. Using Eq. (2), Eq. (37), and Eq. (44), the control function is approximated as

$$u(t)=\frac{1}{b(t)}\left[Mx'(t)+ND^{\nu}x(t)-a\left(t,x(t)\right)\right]=$$
$$\frac{1}{b(t)}\left[M\left(C^{T}F^{\nu}D\phi(t)+x_1-x_0-C^{T}F^{\nu}\phi(1)\right)+NC^{T}\phi(t)\right.$$
$$\left.-a(t,C^{T}F^{\nu}\phi(t)+t\left(x_1-x_0-C^{T}F^{\nu}\phi(1)\right)+x_0)\right], \quad (45)$$

substituting the approximate functions (43) and (45) into the cost functional (1), the following unconstrained optimization problem is obtained

$$\min J=\int_0^1 f(t,C^{T}F^{\nu}\phi(t)+t\left(x_1-x_0-C^{T}F^{\nu}\phi(1)\right)+x_0\,\frac{1}{b(t)}[M\left(C^{T}F^{\nu}D\phi(t)+x_1-x_0-C^{T}F^{\nu}\phi(1)\right)+NC^{T}\phi(t)-a(t,C^{T}F^{\nu}\phi(t)+t\left(x_1-x_0-C^{T}F^{\nu}\phi(1)\right)+x_0)])dt, \quad (46)$$

which can be written as

$$\min J=\int_0^1\Psi(t,C)dt, \quad (47)$$

where

$$\Psi(t,C)=f(t,C^{T}F^{\nu}\phi(t)+t\left(x_1-x_0-C^{T}F^{\nu}\phi(1)\right)+x_0,$$
$$\frac{1}{b(t)}[M\left(C^{T}F^{\nu}D\phi(t)+x_1-x_0-C^{T}F^{\nu}\phi(1)\right)+NC^{T}\phi(t)$$
$$-a(t,C^{T}F^{\nu}\phi(t)+t\left(x_1-x_0-C^{T}F^{\nu}\phi(1)\right)+x_0)]). \quad (48)$$

In order to approximate the integral (47), we use the Legendre-Gauss quadrature rule

$$J[C]\simeq\frac{1}{2}\sum_{j=0}^{l}w_j\Psi\left(\frac{\tau_j+1}{2},C\right), \quad (49)$$

where τ_j and w_j for $j=0,1,\ldots,l$ are $l+1$ zeros of the Legendre polynomial and the corresponding weight, respectively [27]. To find the unknown coefficients $C=[c_1,c_2,\ldots,c_n]$, the following necessary conditions of optimality should be satisfied

$$\frac{\partial J}{\partial c_i}=0, \quad i=0,1,\ldots,n. \quad (50)$$

In this manner, FOCPs (1)-(3) are reduced to a system of algebraic equations (50), which can be solved by Newton's iterative method, and then the approximate functions x(t) and u(t) are obtained by (43) and (45), respectively.

It should be noted that, if the control function cannot be directly derived from Eq. (2), we must approximate it in terms of the Gegenbauer polynomials with unknown coefficients. Then by substituting the approximated state and control functions in the cost functional and applying the optimality conditions, we can obtain the unknown coefficients.

VI. THE CONVERGENCE OF THE METHOD

In the previous section, a new technique for solving a class of FOCPs based on shifted Gegenbauer polynomials was provided. In this section, the convergence of the proposed method is investigated. It is shown that by increasing the number of shifted Gegenbauer polynomials, the approximation error approaches zero. The following lemma presents the upper bound of the error by considering that the function g is $n+1$ times continuously differentiable.

Theorem 2: Consider the function $g:[0,L]\to R, g\epsilon C^{n+1}[0,1]$. Suppose that g_n is the best approximation of g toward Z, then

$$|g-g_n|| \le \frac{H}{(n+1)!}\sqrt{\frac{L^{3+2n}}{3+2n}} \quad (51)$$

where $H = max_{x\in[0,L]}|g^{n+1}(x)|$

Proof: Taylor polynomial of g at zero is defined by

$$z_1(t) = g(0)+g'(0)t+\cdots+g^n(0)\frac{t^n}{n!}, \quad (52)$$

and the upper bound of the error of the Taylor polynomial is written as

$$\left|\left|g(t)-z_1(t)\right|\right| \le \frac{Ht^{n+1}}{(n+1)!}, \quad \forall t\in[0,1] \quad (53)$$

where $H = max_{x\in[0,L]}|g^{n+1}(x)|$. Since $g_n \in Z$ is the best approximation of g out of Z and $z_1 \in Z$, then

$$\left|\left|g-g_n\right|\right|_2^2 \le \int_0^L \left(g(t)-z_1(t)\right)^2 dt \le \int_0^L \left(\frac{Ht^{n+1}}{(n+1)!}\right)^2 dt$$
$$= \left(\frac{H}{(n+1)!}\right)^2 \int_0^L t^{2n+2}dt = \left(\frac{H}{(n+1)!}\right)^2 \frac{L^{3+2n}}{3+2n}, \quad (54)$$

By taking the square roots, the upper bound is achieved.

Corollary1: From Theorem 2, we can conclude that since $g\epsilon C^{n+1}[0,L]$, then $\lim_{n\to\infty} g_n = g$.

Theorem 3: If $g_n(t)$ is the nth approximation of the function $g\epsilon L^2_{\omega_L}[0,L]$, and

$$K_n(g) = \int_0^L [g(x)-g_n(x)]^2 \omega_L^{(a,b)}(x)dx \quad (55)$$

Then we have

$$\lim_{n\to\infty} K_n(g) = 0 \quad (56)$$

Proof: See [28].

Let e_{I^ν} denote the error value for the operational matrix of integration in a Riemann-Liouville sense achieved by using $(n+1)$ orthogonal polynomials in $L^2_\omega[0,1]$, then

$$e_{I^\nu} = F^\nu\phi(t) - I^\nu\phi(t) \quad (57)$$

where $e_{I^\nu} = [e_{I^\nu,0}, e_{I^\nu,1}, \dots, e_{I^\nu,n}]^T$ is an error vector The term $t^{j+\nu}$ based on shifted Gegenbauer polynomials is approximated as

$$t^{j+\nu} \approx \sum_{r=0}^n \tilde{\beta}_r C_{S,r}^{(\nu)}(t). \quad (58)$$

Now by applying Theorem 2 for the term $t^{j+\nu}$, we have

$$||t^{j+\nu} - \sum_{r=0}^n \tilde{\beta}_r C_{S,r}^{(\nu)}(t)||_2 \le \left(\frac{H}{(n+1)!}\right)^2 \frac{L^{3+2n}}{3+2n} \quad (59)$$

and the error norm is calculated as

$$||e_{I^\nu}||_2 = \left\| I^\nu C_{S,i}^{(\beta)}(t) - \sum_{r=0}^i \sum_{j=0}^n \theta_{i,r,j} C_{S,j}^{(\beta)}(t) \right\| =$$
$$|| \sum_{j=0}^i \frac{\Gamma\left(\beta+\frac{1}{2}\right)\Gamma(i+j+2\beta)}{(i-j)!\,j!\,\Gamma\left(j+\beta+\frac{1}{2}\right)\Gamma(2\beta)(-1)^{i-j}} I^\nu(t^j) -$$
$$\sum_{r=0}^i \sum_{j=0}^n \frac{(-1)^{i-r}\Gamma\left(\beta+\frac{1}{2}\right)\Gamma(i+r+2\beta)}{\Gamma\left(r+\beta+\frac{1}{2}\right)\Gamma(2\beta)(i-r)!\,\Gamma(\nu+r+1)} \tilde{\beta}_j C_{S,j}^{(\beta)}(t)||$$

$$\le \sum_{r=0}^n \omega_{ij}^\nu \left|\left| t^{j+\nu} - \sum_{r=0}^n \tilde{\beta}_r C_{S,r}^{(\beta)}(t) \right|\right|_2$$
$$\le \sum_{r=0}^n \omega_{ij}^\nu \left(\frac{H}{(n+1)!}\right)^2 \frac{L^{3+2n}}{3+2n}, \quad (60)$$

Note that $\theta_{i,r,j}$ has been defined as $\theta_{i,r,j} = \sum_{r=0}^n \tilde{\beta}_r \omega_{ij}^\nu$, for $i,j = 0,1,\dots,n$. Therefore, we provide an error upper bound for the fractional Riemann-Liouville operational matrix of shifted Gegenbauer polynomials as:

$$||e_{I^\nu}||_2 \le \sum_{r=0}^n \omega_{is}^\nu \left(\frac{H}{(n+1)!}\right)^2 \frac{L^{3+2n}}{3+2n}. \quad (61)$$

Using Theorems 2 and 3 and Eq. (61), it is concluded that by increasing the number of shifted Gegenbauer basis, the error norm approaches zero. The set of shifted Gegenbauer polynomials forms a basis for the space

$C^n[0,1] =$
$\{f(t)|f$ is continuously differentiable on the interval $[0,1]\}$
with the uniform norm defined by

$$||x||_n = ||x||_\infty + ||x'||_\infty + \cdots \left|\left|x^n\right|\right|_\infty. \quad (62)$$

Let Y_m be a subspace of Banach space $(C^n[0,1], ||.||_n)$ generated by the following shifted Gegenbauer polynomials

$$Y_m = span < C_{S,0}^{(\nu)}(t), C_{S,1}^{(\nu)}(t), \dots, C_{S,n}^{(\nu)}(t) >. \quad (63)$$

Thus, an arbitrary element of Y_m is in the form

$$\eta_0 C_{S,0}^{(\nu)}(t) + \eta_1 C_{S,1}^{(\nu)}(t) + \cdots + \eta_n C_{S,n}^{(\nu)}(t), \quad (64)$$

where η_0, $\eta_1, \dots, \eta_n$ are arbitrary real numbers. Furthermore, on each set Y_m the functional J leads to the function

$$J\left[\eta_0 C_{S,0}^{(\nu)}(t) + \eta_1 C_{S,1}^{(\nu)}(t) + \cdots + \eta_n C_{S,n}^{(\nu)}(t)\right] \quad (65)$$

of n variables η_1, $\eta_2, \dots, \eta_n$. we find η_1, $\eta_2, \dots, \eta_n$ in such a way to minimize the cost functional J. The continuity of the performance index J on the defied space is provided by the following theorem.

Theorem 4: The performance index $J: C^n[0,L]\to R$ given in Eq. (46) is uniformly continuous on the Banach space $(C^n[0,1], ||.||_n)$.

Proof: By the definition of the Riemann-Liouville integral, we have

$$I^\nu g(t) = \frac{1}{\Gamma(\nu)}\int_0^t (t-y)^{\nu-1} g(y)dy \quad (66)$$

$$||I^\nu g(t)||_\infty \le \frac{||g||_\infty}{\Gamma(\nu+1)} \quad (67)$$

$\epsilon > 0$ is given. Suppose that $g(t)\epsilon C^n[0,L]$ and $\delta > 0$. We take $m(t)\,\epsilon\, C^n[0,1]$ such that

$$||g-m||_n \le \sum_{i=0}^n \left|\left|g^i - m^i\right|\right|_\infty < \delta \quad (68)$$

By considering Eq. (67), we have

$$||I^\nu g(t) - I^\nu m(t)||_\infty \le \frac{||g^n - m^n||_\infty}{\Gamma(\nu+1)} \le \frac{\delta}{\Gamma(\nu+1)} \quad (69)$$

According to our assumption on FOCP (1), $f(t,x,u)$ is a continuous function, so for sufficiently small values of $\delta > 0$,

$$|| f(t,g(t),\frac{1}{b(t)}\left(Mg'(t)+ND^{v}g(t)-a(t,g(t))\right) \quad (70)$$
$$-f(t,m(t),\frac{1}{b(t)}(Mm'(t) \ +ND^{v}m(t)-a(t,m(t)))||_{\infty}<\epsilon,$$

provided that $||g-f||<\delta$. Therefore, we obtain

$$||J(g(t))-J(m(t))||<\epsilon. \quad (71)$$

Now, suppose that μ_n is the optimum of the functional J on the subspace Y_m of the Banach space $(C^n[0,1],||.||_n)$.

The next theorem shows the convergence of the proposed method.

Theorem 5: Let μ be the optimum of the functional J on $C^n[0,1]$, then

$$\lim_{n\to\infty}\mu_n = \mu \quad (72)$$

Proof: See [12]

VII. ILLUSTRATIVE EXAMPLES

In this section, we will provide some examples by using the presented scheme. The numerical results will compare to analytical solutions or some other similar schemes in terms of accuracy.

Example 1: Consider the FOCP [21, 22]

$$\min J = \int_0^1 [t^2u^2(t)-(v+2)^2x^2(t)]^2dt$$

subject to the dynamical system

$$x'(t)+D^{v}x(t)=u(t)+t^2$$
$$x(0)=0, \qquad x(1)=\frac{2}{\Gamma(3+v)}.$$

Table I

ABSOLUTE ERROR FOR $x(t)$ OF THE PRESENT METHOD AND THE RESULTS OF [21] WITH v = 1.7 IN EXAMPLE 7.1

	Present method		Method used in [21]	
t	n=2	n=7	n=2	n=7
0.1	3.9e-03	3.3459e-08	9.31e-03	1.15e-6
0.2	5.92e-04	-2.1566e-06	1.27e-02	6.25e-06
0.3	1.4e-03	3.4198e-06	1.23e-02	1.02e-05
0.4	6.79e-04	3.4928e-06	9.89e-03	1.23e-05
0.5	4.5e-03	3.3671e-06	6.82e-03	1.20e-05
0.6	8.9e-03	3.3406e-06	4.06e-03	1.01e-05
0.7	1.23e-02	2.9359e-06	2.13e-03	8.74e-06
0.7	1.32e-02	2.0678e-06	1.12e-03	6.98e-06
0.9	9.8e-03	1.4861e-06	6.87e-04	3.37e-06

Table II

ABSOLUTE ERROR FOR $u(t)$ OF THE PRESENT METHOD AND THE RESULTS OF [22] WITH v = 1.7 IN EXAMPLE 7.1

	Present method		Method used in [22]	
t	n=2	n=7	n=5	n=7
0.1	1.42e-01	2.6412e-04	2.26529e-03	5.61726e-05
0.2	9.02e-02	7.4883e-05	2.52298e-03	2.99579e-05
0.3	4.32e-02	4.0204e-05	3.48464e-03	9.42639e-06
0.4	3.40e-03	3.1686e-05	2.41502e-03	5.26797e-06
0.5	2.71e-02	2.5444e-05	7.42558e-04	9.73733e-06
0.6	4.63e-02	2.0457e-05	5.49078e-04	7.75208e-06
0.7	5.24e-02	1.5456e-05	9.62006e-04	4.64625e-06
0.7	4.36e-02	9.6630e-06	5.09244e-04	6.49362e-07
0.9	1.18e-02	6.0227e-06	2.68469e-04	1.64844e-05

Table III

APPROXIMATE VALUE OF FUNCTIONAL J^* IN EXAMPLE 7.1

J	n=2	n=7
Present Method	4.58×10^{-5}	5.77×10^{-16}
Method used in [21]	2.18×10^{-4}	4.41×10^{-13}

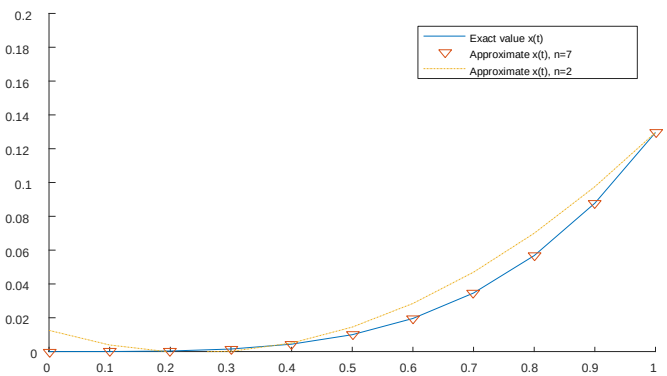


Fig. 1. Exact and approximate solutions of the state function $x(t)$ for n = 2, and n = 7 with v = 1.7 in Example 7.1.

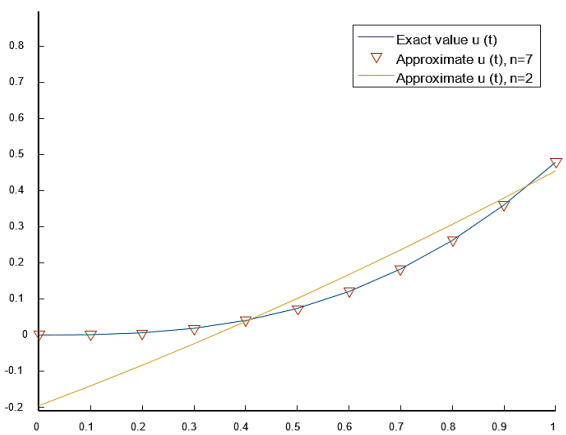


Fig. 2. Exact and approximate solutions of the control function

$u(t)$ for n = 2, and n = 7 with ν = 1.7 in Example 7.1

For the above problem, the exact solutions for the state and control functions are given as

$$x(t) = \frac{2t^{\nu+2}}{\Gamma(\nu+3)},$$

and

$$u(t) = \frac{2t^{\nu+1}}{\Gamma(\nu+2)}$$

the minimum value for the cost functional is $J = 0$. To implement the numerical technique by using the suggested approach, we employ theorem 1 to compute the elements of the operational matrix. Then, we apply Eq. (39) and Eq. (42) to approximate the state and control functions. Finally, by substituting the approximated values in the cost functional and imposing the necessary condition for optimization the unknown coefficients will be calculated. The obtained operational matrix for $\nu = 1.7$ and n = 2 is as follows:

$$F^{\nu} = \begin{bmatrix} 0.2398 & 0.3305 & 0.0820 \\ -0.1102 & -0.1071 & 0.0290 \\ 0.0164 & -0.0171 & -0.0346 \end{bmatrix}$$

Table I demonstrates the comparison of the error approximation for $x(t)$ between the Ritz method along with the Bernstein operational matrix applied in [21] and the present method with n = 2, and n = 7. It can be observed that the present method has significantly reduced the error. Also, the comparison of the obtained values for $u(t)$ by the present method and the method used in [22] is shown in Table II. Our results are comparable with those of the method applied in [22], which used Genocchi polynomials. Table III shows the of J achieved by the present method and the method in [21] for n = 2, and n = 7 with $\nu = 1.7$. The cost functional obtained in this paper is improved in comparison with that of [21]. In addition, the state and control functions for the values of n = 2, n = 7, and $\nu = 1.7$ along with the exact solutions are shown in Fig. 1 and 2, respectively. Based on these figures, it is clear that when the number of terms in the approximations increases, the numerical solutions approach the exact values.

Example 2: Consider the following problem [16, 20]:

$$\min J = \int_0^1 \left(u(t) - x(t)\right)^2 dt$$

subject to the dynamical system

$$x'(t) + D^{\nu}x(t) = u(t) - x(t) + \frac{6t^{\nu+2}}{\Gamma(\nu+3)} + t^3$$

$$x(0) = 0, \quad x(1) = \frac{6}{\Gamma(\nu+4)}.$$

The exact solutions for the state and control functions are given by

$$x(t) = \frac{6t^{\nu+3}}{\Gamma(\nu+4)},$$

and

$$u(t) = \frac{6t^{\nu+3}}{\Gamma(\nu+4)}$$

Table IV demonstrates the comparison of the approximated value for cost functional J between the method in [20] and the present method with n = 5, ν = 1.5, 1.8, and the exact solution. The performance index obtained in this paper is improved in comparison with the results of [20]. In Fig. 3 and Fig. 4, the exact and approximate state functions, as well as the control function, for different values of ν and n = 5 are plotted.

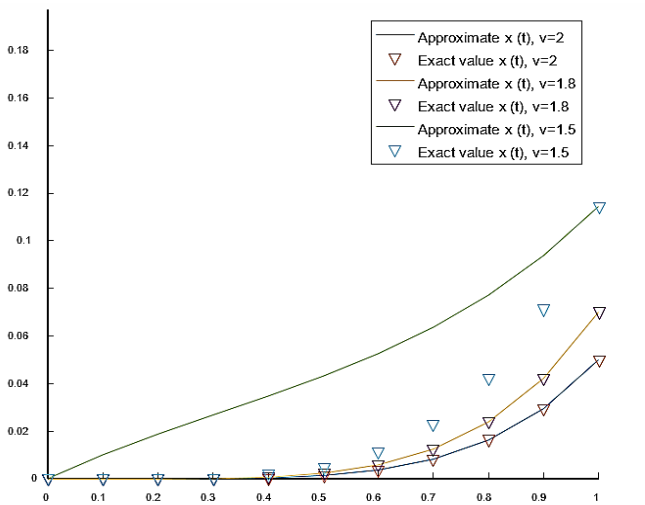


Fig.3. Comparison of the approximate and exact solutions of x(t) for ν= 1.5, 1.8, 2 and n = 5 in Example 7.2

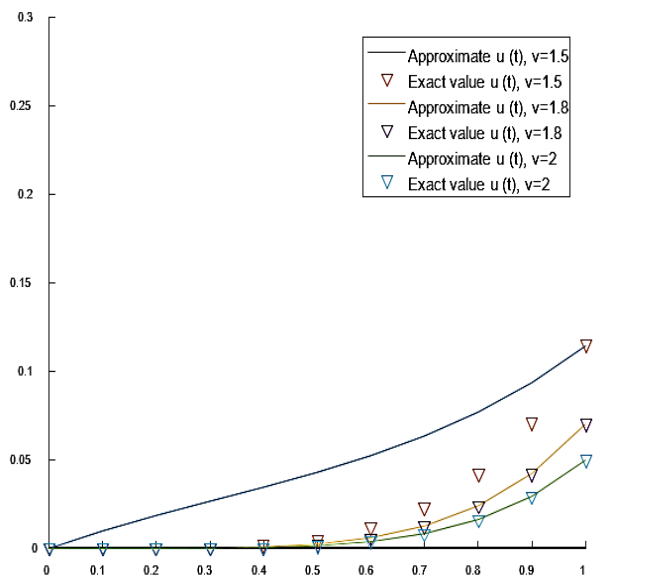


Fig.4. Comparison of the approximate and exact solutions of u(t)for ν = 1.5, 1.8, 2 and $n = 5$ in Example 7.2.

Table IV
APPROXIMATE VALUE OF COST FUNCTIONAL J^* WIIH $n = 5$ FOR EXAMPLE 7.2

J	$\nu = 1.5$	$\nu = 1.8$
Exact value	0	0
Present method	4.84×10^{-12}	4.57×10^{-13}
Method used in [20]	8.5×10^{-11}	1.1×10^{-11}

Example 3: Consider the FOCP [22,29]

$$\min J = \int_0^1 e^t(x(t) - t^4 + t - 1)^2 + (t^2 + 1)(u(t) + 1 - t + t^4 - \frac{8000t^{\frac{21}{10}}}{77\left(\frac{1}{10}\right)})^2 dt$$

subject to the dynamical system

$$D^{1.9}x(t) = u(t) + x(t)$$
$$x(0) = 1, \quad x(1) = 1.$$

The exact solutions of x(t) and u(t) are given by

$$x(t) = 1 - t + t^4,$$

$$u(t) = -1 + t - t^4 + \frac{8000t^{\frac{21}{10}}}{77\left(\frac{1}{10}\right)}$$

The minimum value for the cost functional is $J = 0$. This example is numerically solved by the provided method for different values of n = 5, n = 6, n = 8, and $\nu = 1.9$. The value of cost functional by applying the present method, and the methods in [22, 29] for n = 5,6,8 with $\nu = 1.9$ and the exact value are shown in Table V. The value obtained for J by the suggested numerical scheme is more accurate than the values obtained by the methods in [22, 29]. The absolute errors of the control and state functions for different values of n and $\nu = 1.9$ are plotted in Fig. 5 and 6.

Fig. 7 and 8 display the plot of the approximate and exact, state and control functions for n = 5, 6, 8 and $\nu = 1.9$.

Table V
APPROXIMATE VALUE OF COST FUNCTIONAL J^* IN EXAMPLE 7.3

n	n=5	n=6	n=8
Exact value	0	0	0
Present method	6.78×10^{-8}	1.2×10^{-8}	7.8×10^{-10}
Method in [22]	2.062×10^{-7} 6.7775×10^{-8}	4.8×10^{-8}	2.9×10^{-9}
Method in [29]		_	8.22283×10^{-10}

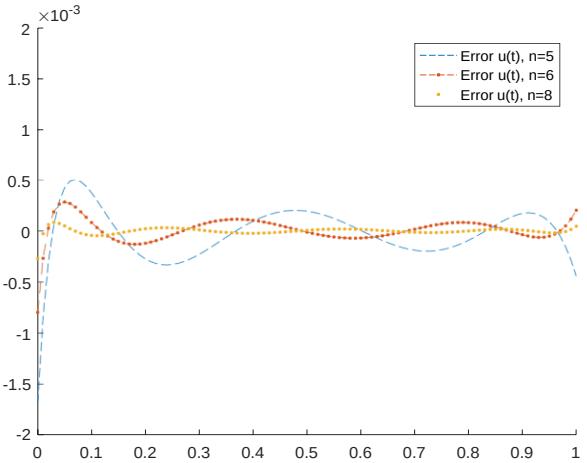


Fig. 5. Comparison of the errors of $u(t)$ for n = 5, 6, 8 and $\nu = 1.9$ in Example 7.3.

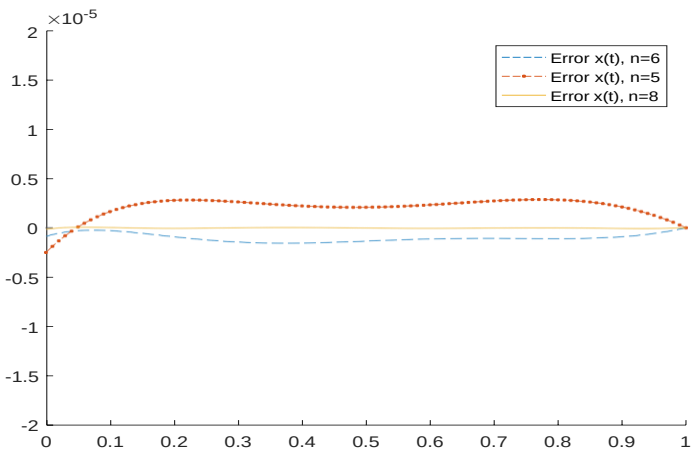


Fig. 6. Comparison of the errors of $x(t)$ for n =5, 6, 8 and ν= 1.9 in Example 7.3.

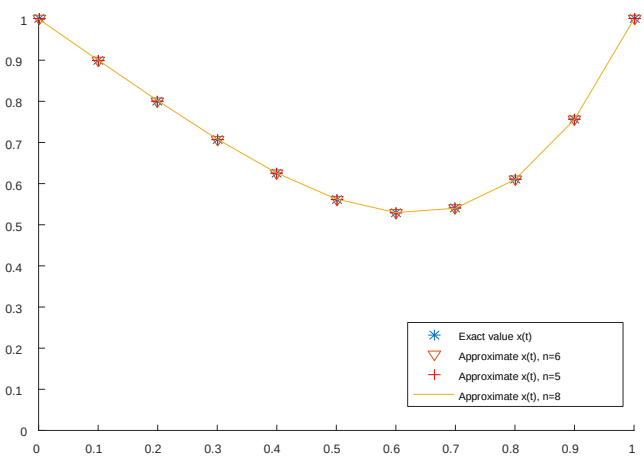


Fig. 7. Comparison of the exact and approximate solutions of $x(t)$ for n = 5, 6, 8 and $\nu = 1.9$ in Example 7.3.

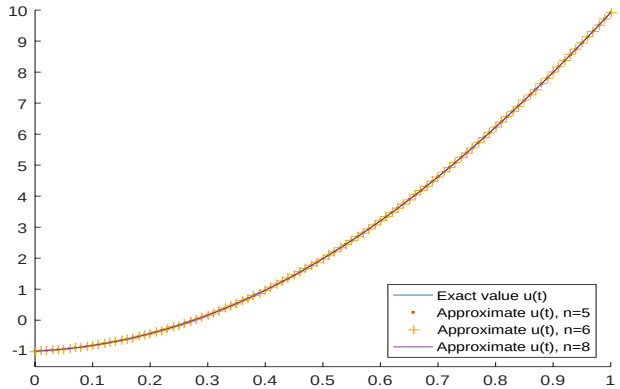


Fig. 8. Comparison of the exact and approximate solutions of $u(t)$ for n =5, 6, 8 and $\nu = 1.9$ in Example 7.3.

VIII. CONCLUSIONS

In this paper, we developed a new numerical scheme for solving a class of FOCPs based on the shifted Gegenbauer polynomials. The shifted Gegenbauer polynomials along with the operational matrix of fractional Riemann-Liouville integration and the Caputo fractional derivative were used to approximate the unknown state and control functions. Thereupon, an unconstrained optimization problem was achieved. Next, by imposing the necessary conditions of optimality, we obtained a system of algebraic equations. Furthermore, we extensively discussed the convergence of the proposed technique. At the end, three illustrative examples were solved and compared with the results in the literature to demonstrate the effectiveness and applicability of the method.

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Farzaneh Soufivand received the B.S. degree in Pure Mathematics from Lorestan University, Lorestan, Iran, in 200٤ and M.S. degree in Applied Mathematics from the Yazd University, Yazd, Iran, in 2006, she joined the Islamic Azad University as an educator in the Department of Mathematics, Doroud Branch, Doroud, Iran. Her current research interests include fractional optimal control problems and fractional differential equations.



Fahimeh Soltanian received the B.S. degree from the Faculty of Mathematics, Isfahan University of Technology, Isfahan, Iran, in 2000, the M.S. and Ph.D. degrees in Applied Mathematics from the Yazd University, Iran, in 2003 and 2009, respectively. Currently she is an Assistant Professor at Department of Mathematics of Payame Noor University, Iran. Her research interests are numerical analysis, optimal control, differential-algebraic equations and numerical optimization.



Kamal Mamehrashi received his B.S. degree in Pure Mathematics from Kharazmi University, Tehran in 2000, his M.S. degree in Applied Mathematics from IASBS, Zanjan in 2003, and his Ph.D. in Applied Mathematics (Control & Optimization) from PNU, Tehran in 2016 under the supervision of Professor Sohrab Ali Yousefi at the University of Shahid Beheshti, Tehran, Iran. He joined the Department of Mathematics at the University of Payame Noor in 2005 as a Lecturer and was promoted to Assistant Professor in 2016. His current research interests include (2D, delay) optimal control problems, (2D, delay) fractional optimal control problems and numerical methods.