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A Discussion on the Existence of Smooth Lyapunov Functions for Continuous Stable Systems

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ABSTRACT

Lyapunov's theorem is the basic criteria to establish the stability properties of the nonlinear dynamical systems. In this method, it is a necessity to find the positive definite functions with negative definite or negative semi-definite derivative. These functions that named Lyapunov functions, form the core of this criterion. The existence of the Lyapunov functions for asymptotically stable equilibrium points is guaranteed by converse Lyapunov theorems. On the other hand, for the cases where the equilibrium point is stable in the sense of Lyapunov, converse Lyapunov theorems only ensure non-smooth Lyapunov functions. In this paper, it is proved that there exist some autonomous nonlinear systems with stable equilibrium points that despite stability don't admit convex Lyapunov functions. In addition, it is also shown that there exist some nonlinear systems that despite the fact that they are stable at the origin, but do not admit smooth Lyapunov functions in the form of $V(x)$ or $V(t,x)$ even locally. Finally, a class of non-autonomous dynamical systems with uniform stable equilibrium points, is introduced. It is also proven that this class do not admit any continuous Lyapunov functions in the form of $V(x)$ to establish stability.

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I. Introduction

In this paper, we focus on two main classes of continuous time nonlinear dynamical systems (NDSs). The first class is described by [1]

$$\dot{x} = f(x), \quad x(0) = x_0 \tag{1}$$

where $D \subset \mathbb{R}^n$, and $f: D \to \mathbb{R}^n$ is a locally Lipschitz function with origin being the equilibrium point (i.e,. $f(0) = 0$).

The second class is described by

$$\dot{x} = f(t,x), \quad x(t_0) = x_0 \tag{2}$$

where $D \subset \mathbb{R}^n$, and $f: [0,\infty] \times D \to \mathbb{R}^n$ is locally Lipschitz in x on $[0,\infty] \times D$ and piecewise continuous in t.

Again, the origin is the equilibrium point of (2) (i.e., $f(t,0) = 0$).

Nonlinear dynamical systems appear in many practical applications including control engineering [2], biological systems [3] and population dynamics [1]. Therefore, the study of stability properties of equilibrium points has always been a crucial issue for mathematicians and control scientists. Among all of this application, stability analysis has attached more attention due to its essential role in real world application including controller design [4], estimation domain of attraction [5] and hybrid systems [6].

It is well known that the origin of (1) is stable in the sense of Lyapunov if

$$\text{for all } \varepsilon > 0, \text{ there exist } \delta = \delta(\varepsilon) > 0, \quad s.t. \ \|x_0\| < \delta \Rightarrow \|x(t)\| < \varepsilon \tag{3}$$

In addition, for the nonlinear system in (2), the equilibrium point ($x = 0$), is stable if

$$\text{for all } \varepsilon > 0, \text{ there exist } \delta = \delta(t_0,\varepsilon) > 0, \quad s.t. \ \|x_0\| < \delta \Rightarrow \|x(t)\| < \varepsilon \ \ t \geq t_0 \tag{4}$$

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and it is uniformly stable when

$$\text{for all } \varepsilon > 0, \text{ there exist } \delta = \delta(\varepsilon) > 0, \quad s.t. \ \|x_0\| < \delta \Rightarrow \|x(t)\| < \varepsilon \quad t \geq t_0 \tag{5}$$

In fact, if the nonlinear system is uniformly stable, then δ does not explicitly depend on t_0.

It should be mentioned that the nonlinear system (2) in its domain $D \subset \mathbb{R}^n$ including origin $x = 0$, is stable in the sense of Lyapunov if there exist a continuously differentiable function $V(t,x): [0,\infty] \times D \to \mathbb{R}$ that satisfies the following conditions for all $t \geq t_0$ and for all $x \in D$

$$V(t,0)=0, \quad V(t,x) \geq \eta_1(x) \tag{6}$$

$$\frac{\partial V}{\partial x} f(t,x) + \frac{\partial V}{\partial t} \leq 0 \tag{7}$$

where, $\eta_1(x)$ is a continuous positive definite function. Also, if $V(t,x)$ satisfies the following inequality

$$V(t,x) \leq \eta_2(x) \tag{8}$$

Then, $x = 0$ is a uniformly stable equilibrium point [7].

It is well known that finding these scalar positive definite functions forms the core of Lyapunov's method. Introducing and computing such suitable Lyapunov functions for NDSs has been one of the main challenges in the literature of NDSs [8-10] and has attracted much attention [11]. In addition, Lyapunov functions are applied in various controller designs [12],[13].

For stable nonlinear systems, it is natural to search for Lyapunov functions to infer stability properties. In fact, the existence of Lyapunov functions are guaranteed by *converse Lyapunov theorems (CLT)* [14, 15]. The study of Lyapunov functions and their properties has become widely popular for researchers in the fields of dynamical systems. For example, for the nonlinear system (1) it is proved in [16] that if the origin is stable in the sense of Lyapunov, and $f(x)$ admits the assumptions in [16] then it is possible to construct some semi-continuous weak Lyapunov functions. The necessary and sufficient conditions for the existence continuous Lyapunov functions can be found in [17].

Recently, special attention has been given to prove the existence of Lyapunov functions for the conditions that the nonlinear system is globally asymptotically stable. For instance, Ahmadi and Kristic in [18] proposed a globally asymptotically stable polynomial vector field with no polynomial Lyapunov functions. Also, Ahmadi and Khadir in [19] investigated a two-dimensional polynomial vector field with no analytic Lyapunov functions. Although in [20], it is proved for the homogeneous vector field of a nonlinear dynamical system that asymptotic stability ensures the existence of a rational Lyapunov function.

Due to the great importance of finding Lyapunov functions with proper analytic properties, this paper has been dedicated to the investigation of stable systems that do not admit smooth Lyapunov functions.

To the best of the author's knowledge, no further results have been available in the context of stable systems. The main contributions of this paper are as follows:

- As a first result, it is proved that if the equilibrium point of (1) is stable, then application of convex Lyapunov functions may fail to satisfy Lyapunov's theorem.
- It is also proved that using smooth Lyapunov functions for the equilibrium point of (1) fails to infer stability properties.
- Finally, it is shown that there exist no smooth Lyapunov functions to establish stability properties of (2).

The continuation of this paper is organized as follows. In Section II, the existence of smooth Lyapunov functions for autonomous stable systems is investigated. Section III is dedicated to some examples of non-autonomous systems that do not admit smooth Lyapunov functions. Simulation results are given in section IV, and section VI concludes the paper.

II. Existence of Smooth Lyapunov Functions for Autonomous Stable Systems

In this section, we propose some examples of autonomous nonlinear dynamical systems and prove that despite stability, they do not admit smooth Lyapunov functions.

Lemma 1. Suppose $V(t,x)$ is a convex Lyapunov function candidate. Then, in the neighborhood of origin, $\nabla V(t,x)^T x$ is positive definite.

Proof. Obviously, for any convex function it holds that

$$V(t, \lambda x + (1-\lambda)y) \leq \lambda V(t,x) + (1-\lambda)(t,y) \tag{9}$$

Rewriting above equation as $\lambda \to 0$ and $y = 0$, implies that $\nabla V^T(t,x) \ x$ is positive definite.

Theorem 1. There exist some nonlinear autonomous systems that despite stability do not admit any convex Lyapunov functions.

Proof . Let the function $f(x)$ in (1) be defined by

$$f(x) = Ax + x\|x\|^{2k} sin^{2m+1}(\frac{1}{\|x\|}) \tag{10}$$

where $x = [x_1 \ x_2]^T$ and all the eigenvalues lie on the imaginary axis $(\pm j\omega_1)$.

This system is stable; however, there are no convex Lyapunov functions in the form $V(x)$ or $V(t,x)$ in order to infer stability.

Due to the fact that all of the eigenvalues of A lie on the imaginary axis, according to linear systems theory, the linear system $\dot{x} = Ax$ is stable in the sense of Lyapunov. Therefore, there exist the positive definite matrix P such that $A^T P +$

$PA = 0$. In order to prove stability of (10), consider $V(x) = x^T Px$, the time derivative of this function along the trajectories of system implies that

$$\dot{V} = 2x^T Px\|x\|^{2k} sin^{2m+1}(\frac{1}{\|x\|}) \quad (11)$$

It is clear that the origin is surrounded by a countable set of limit cycles. To satisfy the condition $\| x \| \le \varepsilon$, it is sufficient to choose $\delta \le \frac{1}{n\pi}$ with large enough values of n.

Now, suppose for the sake of contradiction that there exist a convex Lyapunov function in the form of $V(t,x)$ that satisfies the condition of Lyapunov's theorem.

According to Lyapunov's theorem,

$$\dot{V} = \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial t} \le 0 \quad (12)$$

On the other hand, for any square matrix A, there exist a similarity transformation T such that $T^{-1}AT$ is diagonal. Substitution of $x = Tz$, in (10) leads to

$$\dot{z} = T^{-1}ATz + z\|Tz\|^{2k}\sin(\frac{1}{\|Tz\|})^{2m+1} \quad (13)$$

Now, equation (12) can be written as follows

$$\dot{V} = \frac{\partial V}{\partial z}\frac{\partial z}{\partial x}\dot{x} + \frac{\partial V}{\partial t} = \frac{\partial V}{\partial z}T^{-1}\dot{x} + \frac{\partial V}{\partial t} = \frac{\partial V}{\partial z}\dot{z} + \frac{\partial V}{\partial t} \le 0 \quad (14)$$

Since (14) is held in the region D including origin, it is easy to choose n large enough such that $U = \{r| \frac{1}{n\pi+\frac{3\pi}{4}} \le \|Tz\| \le \frac{1}{n\pi+\frac{\pi}{4}} \ (n = 2k)\}$ is situated in D. Also, due to the fact that $\dot{V}$ is negative definite, $\int_U \dot{V}$ must be negative in this region.

Integrating $\dot{V}$ implies that

$$\int_U \dot{V} = \int_U \left(\frac{\partial V}{\partial z_1}\omega_1 z_2 - \frac{\partial V}{\partial z_2}\omega_1 z_1\right) + \int_U \nabla V^T(t,z)z\|Tz\|^{2k}\sin(\frac{1}{\|Tz\|})^{2m+1} \int_U \frac{\partial V}{\partial t} \quad (15)$$

Let

$$\mathrm{I}_1 = \int_U \left(\frac{\partial V}{\partial z_1}\omega_1 z_2 - \frac{\partial V}{\partial z_2}\omega_1 z_1\right) dS$$
$$\mathrm{I}_2 = \int_U \nabla V^T(t,z)z\|Tz\|^{2k}\sin(\frac{1}{\|Tz\|})^{2m+1} \quad (16)$$
$$\mathrm{I}_3 = \int_U \frac{\partial V}{\partial t}$$

According to Lemma 1, $\nabla V^T(t,z)z$ is positive; as a result, $\mathrm{I}_2 > 0$

Application of divergence theorem implies that

$$\mathrm{I}_1 = \int_U \left(\frac{\partial(V\omega_1 z_2)}{\partial z_1} + \frac{\partial(-V\omega_1 z_1)}{\partial z_2}\right) dS = \int_{\partial U}\left([V\omega_1 z_2 - V\omega_1 z_1].\begin{bmatrix}\frac{z_1}{\sqrt{z_1^2+z_2^2}} \\ \frac{z_2}{\sqrt{z_1^2+z_2^2}}\end{bmatrix}\right) dS = 0 \quad (17)$$

Because of the fact that $\int_U \dot{V} \le 0$ and $I_2 > 0$ in the region U, there exist a region $S_1 \subseteq U$ such that

$$\nabla V^T(t,z)z\|Tz\|^{2k}\sin(\frac{1}{\|Tz\|})^{2m+1} + \frac{\partial V}{\partial t} < 0 \text{ in } S_1 \quad (18)$$

On the other hand, according to Lemma 1, $\nabla V^T(t,z)z$ is positive-definite. Without loss of generality, assume that

$$\nabla V^T(t,z)z \ge V_1(z) \quad (19)$$

Where $V_1(z)$ is positive definite. Substitution of (19) in equation (18) infers that

$$V_1(z)\|Tz\|^{2k}\sin(\frac{1}{\|Tz\|})^{2m+1} + \frac{\partial V}{\partial t} < 0 \text{ in } S_1 \quad (20)$$

This implies that

$$V(t,z) < -V_1(z)\|Tz\|^{2k}\sin\left(\frac{1}{\|Tz\|}\right)^{2m+1} t + V(0,z) \text{ in } S_1 \quad (21)$$

Since S_1 is bounded and the functions $V_1(z)\|Tz\|^{2k}\sin\left(\frac{1}{\|Tz\|}\right)^{2m+1}$ and $V(0,z)$ are continuous in S_1, let

$$\varepsilon_1 = \inf_{S_1} V_1(z)\|Tz\|^{2k}\sin\left(\frac{1}{\|Tz\|}\right)^{2m+1} \quad (22)$$

$$\varepsilon_2 = \mathrm{Sup}_{S_1} V(0,z) \quad (23)$$

Then the inequality in (21) can be written follows

$$V(t,z) < -\varepsilon_1 t + \varepsilon_2 \text{ for all } t > 0 \quad (24)$$

Now, to prove that there are no convex Lyapunov functions in the form of $V(t,z)$, choose $t > \frac{\varepsilon_2}{\varepsilon_1}$. This implies that $V(t,z) < 0$, which is a contradiction because $V(t,z) > 0$ for all $z \ne 0$. With a similar reasoning, it is easy to show that there are no convex Lyapunov functions in the form $V(z)$ to establish stability.

Theorem 2 There exist some nonlinear autonomous systems that despite stability do not admit smooth Lyapunov functions.

Proof. Let $f(.)$, in system (1) be defined by

$$f(x) = \begin{cases} x_2 + x_1(x_1^2 + x_2^2)^2 \sin^2(\frac{1}{\sqrt{x_1^2+x_2^2}}) \\ -x_1 + x_2(x_1^2 + x_2^2)^2 \sin^2(\frac{1}{\sqrt{x_1^2+x_2^2}}) \end{cases} \quad (25)$$

This system is stable in the sense of Lyapunov; however, smooth Lyapunov functions fail to establish stability.

In order to prove stability, consider $V(x) = x_1^2 + x_2^2$, the time derivative of this function along the trajectories yields

$$\dot{V} = 2(x_1^2 + x_2^2)^3 sin^2(\frac{1}{\sqrt{x_1^2 + x_2^2}}) \quad (26)$$

It is clear that the origin is surrounded by a countable set of limit cycles and all of the limit cycles are semi-stable. To ensure that $\| x \| \le \varepsilon$, it is sufficient to choose $\delta \le \frac{1}{n\pi}$ with large enough values of n.

For the sake of contradiction, suppose that there exists a smooth Lyapunov function in the form of $V(t,x)$ which satisfies the Lyapunov's theorem. According to (7),

$$V(t,x(t)) \le V(t_0,x_0) \quad (27)$$

In order to complete the proof, it is sufficient to determine and compare the values of Lyapunov function on the limit cycles. It is noticeable that all trajectories starting from the inside of a limit cycle return to the same cycle whereas the outer trajectories approach to the next one. In fact, if the system starts from the outside of $r = \sqrt{x_1^2 + x_2^2} = \frac{1}{n\pi}$, it eventually converges to $r = \frac{1}{(n-1)\pi}$, thereby causing the value of Lyapunov function on the surface $r = \frac{1}{(n-1)\pi}$ to be less or equal to the value of Lyapunov function on the surface $r = \frac{1}{n\pi}$. This follows from the fact that by starting (25) from the outside of $\frac{1}{n\pi}$, the system remains bounded. Since $V(t,x(t))$ is a decreasing continuous function, it has a limit c as $t \to \infty$. In other words, the value of $V(t,x(t))$ on the surface of limit cycles is constant and $V(t,x_n(t)) \le V(t_0,x_{n-1}(t_0))$ where $r_n = \|x_n\| = \frac{1}{n^2\pi^2}$ and $r_{n-1} = \|x_{n-1}\| = \frac{1}{(n-1)^2\pi^2}$. Consider a_n as a sequence defined by

$$a_n = V(t,r_n) \text{ where } r_n = \frac{1}{n^2\pi^2} \quad (28)$$

In other words, a_n is a sequence for which its value is computed on the surface of limit cycles.

Let $\gamma_1 = \sup V(t,x) \le \sup V_2(x)$, it is clear that the regions described by $r_n \le \frac{1}{n\pi}$ are bounded and $V_2(x)$ is continuous; consequently, γ_1 must be bounded. This implies that a_n is a bounded sequence.

Due to the fact that the value of Lyapunov function on the surface $r_{n-1} = \frac{1}{(n-1)\pi}$ must be less or equal to the value of Lyapunov function on the surface $r_n = \frac{1}{n\pi}$, a_n is an increasing sequence. This means that

$$a_1 \le a_2 \dots \le a_{n-1} \le a_n \quad (29)$$

Because of the fact that a_n is a bounded and increasing sequence, $\lim_{n\to\infty} a_n$ exists. Assuming $n \to \infty$ implies that $a_n \to 0$ because $x_n \to 0$.

According to (29), $a_1 \le a_2 \dots \le a_{n-1} \le 0$ and this yields that $a_1 \le 0$, which contradicts the fact that $V(t,x_1) = a_1$ must be positive for all $x \ne 0$. This shows that there are no smooth Lyapunov functions in the form of $V(t,x)$ to infer stability of (25) .

It is worth noting that for the system (25) , the results of Theorem 2 do not contradict with CLT, because they guarantee non-smooth Lyapunov functions in general. For more explanations, please see [21],[22]. Although there are no smooth Lyapunov functions in the form of $V(x)$ or $V(t,x)$ for (25), there might exist some lower semi-continuous Lyapunov functions according to [23]. In addition, Theorem 2 confirms the results in [24] through presenting some examples to show that the existence of a continuous Lyapunov function for (1) does not infer the existence of a locally Lipschitz Lyapunov function. Also, it shows that the existence of a Lipschitz Lyapunov function does not infer the existence of a continuously differentiable Lyapunov function in general.

The results of Theorem 2 can be extended to another class of nonlinear systems. Theorem 3 is devoted to this topic.

Theorem 3 There exist some n dimensional nonlinear autonomous systems that despite stability do not admit smooth Lyapunov functions.

Proof. For proving this theorem, consider the following nonlinear system which is described by:

$$\begin{aligned} \dot{x_1} &= x_2 + x_1(x_1^2 + x_2^2)^2 \sin^2(\frac{1}{\sqrt{x_1^2+x_2^2}}), \\ \dot{x_2} &= -x_1 + x_2(x_1^2 + x_2^2)^2 \sin^2(\frac{1}{\sqrt{x_1^2+x_2^2}}), \\ \dot{y} &= f(y) \end{aligned} \quad (30)$$

Where the nonlinear system $\dot{y} = f(y)$ is stable in the sense of Lyapunov.

The trajectories of this system i.e., $\eta = [x_1 \; x_2 \; y^T]^T$ are stable in the sense of Lyapunov; however, the task of establishing stability through smooth Lyapunov function fails.

Suppose for the sake of contradiction that there exist a smooth Lyapunov function in the form of $V(t,x,y)$ for which it satisfies the condition of Lyapunov's theorem. Application of Lyapunov's theorem, infers that

$$V(t,x(t),y(t)) \le V(t_0,x_0,y_0) \tag{31}$$

The relation (31) must hold for all $\eta \neq 0$. Suppose $y = 0$, then (31) turns to

$$V(t,x(t),0) \le V(t_0,x_0,0) \tag{32}$$

This implies that for the nonlinear system discussed in Theorem 2, there exist a smooth Lyapunov function in the form of $V(t,x)$ which satisfies the conditions of Lyapunov's theorem and this is a contradiction because it is shown that there are no smooth Lyapunov functions to establish the stability of this system.

During the next theorem it is shown that Theorem 2 could be generalized for proving the existence of the smooth functions, used in LaSalle's invariance principle [25].

Theorem 4 There exist some nonlinear autonomous systems that despite having an infinite number of limit cycles, do not admit continuous functions in the form $V(x)$ to establish LaSalle's invariance principle.

Proof. In the proof of Theorem 2, it is illustrated that there exist numerous limit cycles for (25).

Consider a continuous function $V(x)$ with a minimum at x_p, that satisfies LaSalle's invariance principle [25]. Then, V must be decreasing in the region ψ described by $V(x) \le c$. Since ψ is compact, consider a_1 as follows:

$$a_1 = \min_{\psi} V(x) \tag{33}$$

Suppose that the function $V(x)$ has a global minimum point at x_p, let $a_1 = V(x_p)$, and define

$$W(x) = V(x) - a_1 \tag{34}$$

It is clear that $W(x) \ge 0$. In addition, $W(x)$ and $V(x)$ have the same trend of monotonicity. This means that for the nonlinear system (25) there exist a decreasing continuous positive function $W(x)$ which is not essentially positive definite. The remainder of proof could be followed from the proof of Theorem 2.

In the continuation of this paper, it is proved that there exist a similar criterion for non-autonomous dynamical systems.

III. Existence of Smooth Lyapunov Functions for Nonautonomous Stable Systems

The motivation of this section is to show that the results of Theorem 2 can be generalized for non-autonomous systems.

Theorem 5 There exist some two dimensional nonlinear non-autonomous systems that despite stability do not admit smooth Lyapunov functions in the form of $V(x)$.

Proof. Let the nonlinear system (2) be defined by

$$f(t,x) = \begin{cases} \dfrac{f(t)x_1}{1+{x_1}^2} \\ \dfrac{f(t)x_2}{1+{x_2}^2} \end{cases} \tag{35}$$

Where $f(t) > 0$ and $\int_{t_0}^{\infty} f(\tau)d\tau = M_1 < \infty$. This system is stable but there are no continuous Lyapunov functions in the form of $V(x)$ in order to prove its stability.

Rewriting (35) implies that

$$\frac{dx_1}{x_1} = \frac{f(t)dt}{1+{x_1}^2} \Rightarrow \int_{x_1(0)}^{x_1(t)} \frac{dx_1}{x_1} = \int_{t_0}^{t} \frac{f(\tau)d\tau}{1+{x_1}^2} \le \int_{t_0}^{t} f(\tau)d\tau = M_1 \Rightarrow \ln(|x_1(t)|) - \ln(|x_1(0)|) \le M_1 \Rightarrow \tag{36}$$

$$\ln\left(\frac{|x_1(t)|}{|x_1(0)|}\right) \le M_1 \Rightarrow \sup(|x_1(t)|) < |x_1(0)|e^{M_1} \tag{37}$$

With a similar reasoning, it is easy to see that $\sup(|x_2(t)|) < |x_2(0)|e^{M_1}$. Application of (36) and (37) leads to

$$\|x\|^2 \le x_1(0)^2 e^{2M_1} + x_2(0)^2 e^{2M_1} = \|x_0\|^2 e^{2M_1} \tag{38}$$

For any given ε, in order to satisfy (4) , it is sufficient to choose $\delta \le \varepsilon e^{-M_1}$, because

$$\sup(\|x(t)\|) \le \|x_0\| \ e^{M_1} \Rightarrow \sup(\|x(t)\|) \le \varepsilon e^{-M_1} e^{M_1} = \varepsilon \Rightarrow \|x(t)\| \le \varepsilon \tag{39}$$

Equation (39) infers that, the system (35) is stable in the sense of Lyapunov. Due to the fact that δ does not depend on t_0 explicitly, (35) is uniformly stable (please see equation (5)).

To prove that there are no continuous Lyapunov functions to establish stability, it is sufficient to rewrite (35) in the form of

$$\left(\frac{1}{x_1} + x_1\right)dx_1 = f(t)dt \Rightarrow \ln(|x_1(t)|) - \ln(|x_1(0)|) + \frac{x_1(t)^2}{2} - \frac{x_1(0)^2}{2} = \int_{t_0}^{t} f(\tau)d\tau. \tag{40}$$

If $x_1(0) > 0$, (40) can be written as follows

$$x_1(t)e^{\frac{x_1(t)^2}{2}} = x_1(0)e^{\frac{x_1(0)^2}{2}} e^{\int_{t_0}^{t} f(\tau)d\tau} \tag{41}$$

Suppose $\lim_{t\to\infty} x_1(t) = y_1$, this ensures that

$$y_1 e^{\frac{{y_1}^2}{2}} = x_1(0)e^{\frac{x_1(0)^2}{2}} e^{M_1} \tag{42}$$

Equation (42) implies that y_1 is a function of $x_1(0)$. In addition, since y_1 is an invertible function; for any final state y_1 there exist an initial condition $x_1(0)$ that satisfies (42) . In general, for any final state $y_1(n-1)$ there exist the initial condition $y_1(n)$ such that it satisfies (42). Similar reasoning could be applied to derive (42) for x_2.

For the sake of contradiction, suppose that there exist a smooth Lyapunov function which satisfies the condition of Lyapunov's theorem, then a ,mccording to Lyapunov's theorem, there must exist a decreasing positive definite function in the region D. Consider $\bar{x} = [x_1\ x_2]^T \epsilon D$ where $x_i > 0$. According to previous explanations, there exist the final state $y_1 = \begin{pmatrix} y_1(1) \\ y_2(1) \end{pmatrix}$ which satisfies (42). In general, for any final state y_{n-1} there exists an initial condition y_n that satisfies (42). Due to the fact that $V(x)$ is a decreasing function $V(y_{n-1}) \le V(y_n)$. This relation says that

$$V(y_1) \le V(y_2) \dots \le V(y_{n-1}) \le V(y_n) \quad (43)$$

Let $\lim_{n\to\infty} y_1(n) = L_1$, according to (42),

$$L_1 e^{\frac{L_1{}^2}{2}} = L_1 e^{\frac{L_1{}^2}{2}} e^{M_1} \quad (44)$$

Equation (44) infers that $L_1 = 0$. Due to the fact that $V(x)$ is continuous,

$$\lim_{n\to\infty} V(y_n) = V\left(\lim_{n\to\infty} y_n\right) = V(0) = 0 \quad (45)$$

Rewriting (43) implies that

$$V(y_1) \le V(y_2) \dots \le V(y_{n-1}) \le 0 \quad (46)$$

According to (46), $V(y_1) \le 0$, and this is a contradiction because $V(x)$ must be positive for all $x \ne 0$.

IV. SIMULATION RESULTS

In this section, we present some results to demonstrate the effectiveness of the proposed theorems. In fact, some nonlinear systems are investigated, all of them being stable in the sense of Lyapunov, that do not admit smooth Lyapunov functions.

Example 1 Consider the nonlinear system (10), with $A = \begin{pmatrix} -4 & 17 \\ -1 & 4 \end{pmatrix}$. It is easy to see that the positive definite matrix $P = \begin{pmatrix} 1 & -4 \\ -4 & 17 \end{pmatrix}$ satisfies $A^T P + PA = 0$. The trajectories of this system for the initial value $(0.03,0)$ are illustrated in Figure 1.

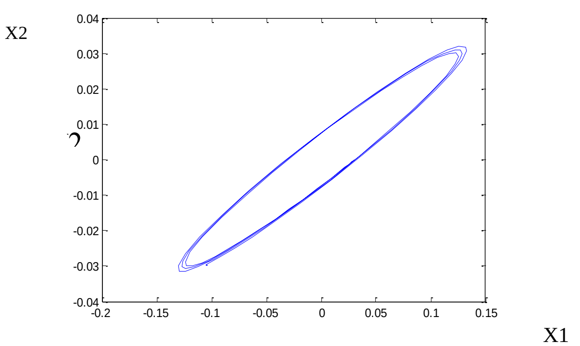

Fig. 1: The trajectories of (10) for initial value $(0.03,0)$.

Fig. **1** shows that the system is stable, although in Theorem 1 it was proved that it does not admit convex Lyapunov functions.

Example 2. Consider the nonlinear system (25) with the initial condition $(0, \frac{-1}{\pi} + 0.1)$.

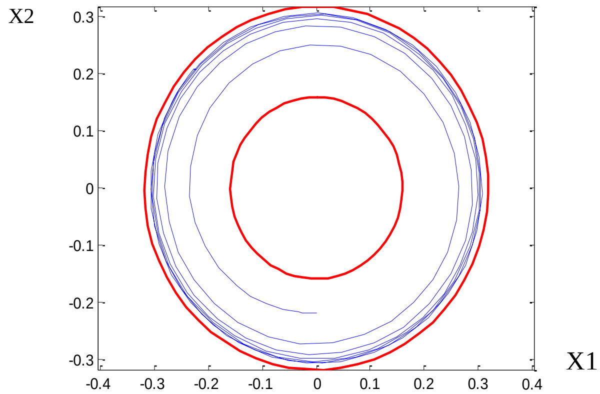

Fig.2: The trajectories of (25) for initial value $\left(0, \frac{-1}{\pi} + 0.1\right)$.

The simulation results confirm that all of the limit cycles for the system are semi-stable. As shown in Figure 2, the innermost curve is a limit cycle described by $\frac{1}{2\pi}$ and the outermost curve is a limit cycle described by $\frac{1}{\pi}$.

Example 3. The purpose of this example is to show the simulation results for the case of non-autonomous dynamical systems. Consider the nonlinear system (35) for $(t) = \frac{1}{t^4+1}$. To satisfy (3) with $\varepsilon = 0.01$, choosing $\delta \le 0.01 e^{-\frac{\pi}{2\sqrt{2}}} = 0.003$ confirms that $\|x\| \le \varepsilon$. The results are depicted in Figure 3.

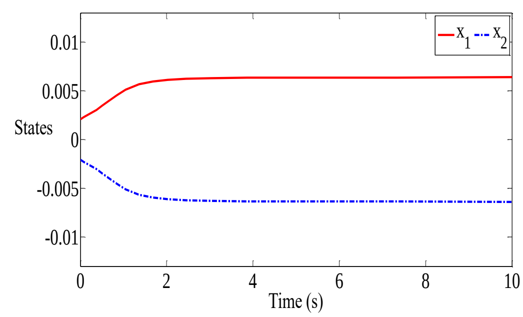

Fig. 3: The trajectories of (35) for the initial condition (0.0021,-0.0021)

V. CONCLUSIONS

This paper studies the stable NDSs that do not admit smooth Lyapunov functions. It is proved that the existence of smooth Lyapunov functions is not necessary for the case that NDSs are stable in the sense of Lyapunov. In fact, the stability analysis of two different classes of nonlinear dynamical systems is discussed and it is shown that their stability cannot be established by any convex differentiable Lyapunov functions or smooth Lyapunov functions.

As a first result, we discussed a NDS which is stable; however, investigating the stability of this system through convex Lyapunov functions is impossible. Subsequently, we introduced a class of NDS and proved that there are no smooth Lyapunov functions to establish the stability through classical Lyapunov method. In addition, a general class of NDSs were investigated and it is shown that they not admit continuous Lyapunov functions in the form $V(x)$ or $V(t,x)$. Furthermore, it is proved that LaSalle's invariance principle is unable to infer the convergence of trajectories towards the invariant sets. Finally, we present a stable non-autonomous dynamical system and prove that there are no smooth Lyapunov functions in the form of $V(x)$ to establish stability. From the second part of this paper, it can be inferred that existence of smooth Lyapunov function in the form of $V(x)$ or $V(t,x)$ is not necessary for stable equilibrium points and third part states that there exist some non-autonomous systems with uniformly stable equilibrium point that don't admit continuous Lyapunov function in the form of $V(x)$.

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