Hybrid Functions to Solve Fractional Optimal Control Problems Using the Collocation Method

Seyed Mehdi Shafiof,† Javad Askari, and Maryam Shams Solary

1,3 Department of Mathematics, Payame Noor University, Tehran, Iran
2 Department of Electrical and Computer Engineering, Isfahan University of Technology, Isfahan, Iran

Abstract

This article aims to introduce a modern numerical method based on the hybrid functions, consisting of the Bernoulli polynomials and Block-Pulse functions. An indirect approach is proposed for solving the fractional optimal control problems (FOCPs). Firstly, the two-point boundary value problem (TPBVP) is calculated for a class of FOCPs, including integer-fractional derivatives, leading to a system of fractional differential equations (FDEs), which have the left and right-sided Caputo fractional derivatives (CFD). Therefore, a new approach is proposing to achieve the left Riemann-Liouville fractional integral (LRLFI) and right Riemann-Liouville fractional integral (RRLFI) operators for Bernoulli hybrid functions. Then, hybrid functions approximation, LRLFI and RRLFI operators, and the collocation method are used to solve the TPBVP. The error bounds for the hybrid function and LRLFI and RRLFI operators are also presented. Moreover, the convergence of the proposed method is proved. Finally, the simplicity and accuracy of the method are illustrated using some numerical examples.

Keywords: Bernoulli polynomials, Block-Pulse functions, Collocation method, Fractional optimal control, Hybrid functions, Riemann-Liouville fractional integral operators

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I. Introduction

In 1695, the idea of fractional calculus was firstly discussed in some correspondence between Leibniz and Hopital. Leibniz predicted that “one day, the useful consequences would be drawn,” although his vision has been realized during the following years. Later, fractional calculus became a fascinating area for mathematicians, and many forms of fractional differential operators were proposed. The most popular ones are the Riemann-Liouville and Caputo fractional derivatives [1]-[4]. Fractional calculus is an effective modeling tool that displays the behavior of many mechanical and biological dynamics better than the integer models [5]. Recently, fractional calculus has been widely used in different fields of applied sciences such as physics, engineering, and bioengineering [6], stochastic systems [7], and the theory of diffusion [8]. One of the latest applications of fractional models is to investigate and analyze the complex transmission pattern of COVID-19 disease [9], [10].

The optimal control problems have been studied in almost many fields to provide broad background knowledge. These problems minimize an objective function, subject to dynamic constraints on the state and control variables [11], [12]. FOCPs are a subclass of optimal control problems, which objective function or constraint equation governed by FDE [13]. Since the dynamic constraints of these problems involve FDE, finding the exact solution is very complicated, so numerical methods must be utilized to solve FOCPs. These numerical methods can be divided into two major classes, including direct and indirect methods. The indirect methods are based on Pontryagin’s maximum principle and the necessary optimality conditions for FOCPs. These conditions lead to a TPBVP that can analytically or numerically be solved using widely known methods for differential equations. For this purpose, Agrawal used the calculus of variation to specify the optimality conditions for FOCPs, based on the Riemann-Liouville fractional derivatives (RLFD) [14]. He also set the optimality conditions for FOCPs containing CFD [15]. The TPBVP,
including left and right fractional derivatives, have approximately been solved by the Legendre multiwavelet collocation method [16], variational iteration and Adomian decomposition methods [17], and fractional power series neural network method [18]. The advantages of indirect methods are explained in [19], [20]. In the direct methods, FOCP is solved by minimizing the objective function, using the approximation or discretization of unknown functions, according to the problem constraints, and without deriving the Hamiltonian equations. Some examples of this method are as follows: Hybrid of Block-Pulse functions and orthonormal Taylor polynomials [21], Bernoulli polynomials with the operational matrix of fractional integration [22], Genocchi operational matrix of integration [23], Fractional-order Bessel wavelet functions [24].

In recent years, many authors have applied hybrid functions to solve different problems because hybrid functions are a mathematical power tool to approximate the functions defined on the distinct subintervals. Hybrid functions have consisted of the combination of a polynomial with piecewise constant basis functions such as Block-Pulse. For more details about this topic, can refer to [25]-[27].

In the current article, we focus on the FOCP characterized by integer-fractional dynamical system as follows:

\[ \min J = \int_{\tau_0}^{\tau_f} f(t, x(t), u(t))dt, \]

subject to

\[ K_1 \dot{x}(t) + K_2 \ddot{x}(t) = g(t, x(t)) + b(t)u(t), \]

with initial condition

\[ x(t_0) = x_0, \quad 0 < \alpha < 1, \]

where \( f(t, x(t), u(t)), g(t, x(t)) \) and \( b(t) \) are smooth, \( b(t) \) is non-zero and, \( K_1 \) and \( K_2 \) are scalar numbers. As a practical example, the dynamical system of light amplification in Erbium-doped fiber amplifier, one of the most commonly applied types of fiber amplifiers in metro optical networks, contains the integer-fractional derivatives [28].

This article presents a new method to obtain the LRLFI, and RRLFI operators for the hybrid of Bernoulli polynomials and Block-Pulse functions directly and without any approximation. The accuracy of these operators is effective in increasing the accuracy of the proposed method for solving FOCPs. We employ the following steps to solve the stated FOCP using the suggested indirect method.

First, TPBVBP based on the left and right CFD is obtained from the necessary optimality conditions of the FOCP. Then, in the obtained FDEs, the highest order derivatives of the state \( x(t) \) and costate \( \lambda(t) \) functions are approximated by using the Bernoulli hybrid functions with unknown coefficients. Afterward, by applying the LRLFI, RRLFI operators, and Legendre-Gauss collocation method, the problem is converted to a system of algebraic equations, by the solution of which the unknown coefficients are obtained. Besides, estimation errors and convergence analysis are brought. The existing results demonstrate that by increasing the number of hybrid functions basis, the approximate solutions converge to the exact solutions.

The organization of this paper is as follows: In Section II, we provide some necessary definitions and properties of fractional calculus and hybrid functions. Section III, introduces LRLFI and RRLFI operators for hybrid functions. In Section IV, we obtain the necessary optimality conditions and describe the numerical method for solving FOCPs. We provide error bound and convergence analysis for the proposed method in Section V. In Section VI, test problems are used to verify the accuracy of the suggested approach. The final section of this article is the conclusion.

II. PRELIMINARIES AND NOTATIONS

We present some basic definitions and some required properties of the fractional calculus and hybrid functions used in this article.

A. Fractional Calculus

In this part, we briefly provide some definitions of the fractional derivative and integral, and their properties [1]-[4], [29]. Let \( f: [a,b] \to \mathbb{R} \) be a function, and \( \alpha > 0 \) be the order of fractional derivative and integral, and \( m = [\alpha] + 1 \).

**Definition 1** The LRLFI and RRLFI operators of order \( \alpha \) are respectively defined as follows:

\[ d^\alpha_I f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau)d\tau, \]

\[ d^\alpha_R f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (t - \tau)^{\alpha-1} f(\tau)d\tau, \]

where \( \Gamma(\alpha) \) is the Gamma function.

**Definition 2** The left and right RLFDO operators of order \( \alpha \) are given by:

\[ D^\alpha_L f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{d^m}{dt^m} f(t) (t - \tau)^{m-\alpha-1} d\tau, \]

\[ D^\alpha_R f(t) = \frac{1}{\Gamma(m-\alpha)} \int_t^b \frac{d^m}{dt^m} f(t) (t - \tau)^{m-\alpha-1} d\tau. \]

**Definition 3** The left and right CFD operators are defined by:

\[ \zeta_a D^\alpha_L f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{d^m}{dt^m} f(t) (t - \tau)^{m-\alpha-1} d\tau, \]

\[ \zeta_b D^\alpha_R f(t) = \frac{1}{\Gamma(m-\alpha)} \int_t^b \frac{d^m}{dt^m} f(t) (t - \tau)^{m-\alpha-1} d\tau. \]

Some properties of the Caputo derivative are as follows:

\[ \zeta_a D^\alpha_L a^\alpha D^\beta_L f(t) = \frac{f(t)}{\Gamma(\alpha - \beta)} - \sum_{i=0}^{m-1} \frac{f^{(i)}(a)}{i!} (t - a)^i, \]

\[ \zeta_a D^\alpha_L K = 0, (K \text{ is constant}), \]

\[ \zeta_0 D^m_L f(t) = \frac{f^{(m)}(t)}{\Gamma(m+1)}, \]

\[ \zeta_0 D^m_R f(t) = (-1)^m \frac{f^{(m)}(t)}{\Gamma(m+1)}, \quad m \in \mathbb{N}, \]

\[ \zeta_0 D^\alpha_L f(t) = a^\alpha D^\alpha_L f(t), \]

\[ \zeta_0 D^\alpha_R f(t) = (-1)^m a^\alpha D^{m-\alpha} f(t), \quad a \notin \mathbb{N}. \]
The relation between RCFD and RRLFD is as follows:

\[ \zeta D^\alpha_a f(t) = \frac{\partial}{\partial t}D^\beta_b f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{\Gamma(k+1-\alpha)}(b-t)^{k-\alpha}. \]  

(14)

The following relations are established:

\[ a^\beta_i a^\alpha_t f(t) = a^\alpha_t a^\beta_i f(t), \quad t^\beta_i t^\alpha_b f(t) = t^\alpha_b t^\beta_i f(t), \quad \alpha, \beta > 0 \]  

(15)

\[ l^\alpha_i l^\beta_b f(t) = f(t) - \sum_{i=0}^{m-1} (-1)^i \frac{l^{(i)}(b)}{i!}(b-t)^i. \]  

(16)

\[ l^\beta_b (b-t)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+1)}(b-t)^{\beta-1}, \quad \alpha, B > 0. \]  

(17)

\[ \alpha l^\beta_t = \frac{\Gamma(r+1)}{\Gamma(r+\alpha)} t^{r+\alpha}, \quad r \in \mathbb{N}, \quad t > 0. \]  

(18)

\[ \| a^\alpha_i f(t) \|_{L^2(a,b)} \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \| f(t) \|_{L^2(a,b)}, \quad 0 < \alpha \leq 1. \]  

(19)

\[ \| t^\beta_i f(t) \|_{L^2(a,b)} \leq \frac{(b-a)^\beta}{\Gamma(\beta+1)} \| f(t) \|_{L^2(a,b)}, \quad 0 < \beta \leq 1. \]  

(20)

LRLF operator satisfies the following formula:

\[ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t), \]  

where \( t^{\alpha-1} \ast f(t) \) is the convolution of \( t^{\alpha-1} \) and \( f(t) \). Also, LRLF and CFD operators are linear operations.

### B. Bernoulli Polynomials

Bernoulli polynomials of order \( m \) can be defined as [30]:

\[ \beta_m(t) = \sum_{k=0}^{m} \binom{m}{k} \alpha_{m-k} t^k, \]  

(22)

where \( \alpha_k = \beta_k(0), k = 0,1,2,...,m \), are Bernoulli numbers. The Bernoulli polynomials satisfy the following formula:

\[ \beta_m(1-t) = (-1)^m \beta_m(t). \]  

(23)

### C. Hybrid of Block-Pulse Functions and Bernoulli Polynomials

Hybrid functions \( b_{nm}(t) \) for \( n = 1,2,...,N, m = 0,1,2,...,M \) are given over the interval \([0,1]\) as:

\[ b_{nm}(t) = \begin{cases} \beta_m(Nt - n + 1), & t \in \left[\frac{n-1}{N}, \frac{n}{N}\right], \\ 0, & \text{otherwise}, \end{cases} \]  

(24)

where \( n \) and \( m \) are the orders of the Block-Pulse functions and Bernoulli polynomials, respectively. It is clear that \( Y = \text{span}\{b_{10}(t), b_{20}(t), ..., b_{NO}(t), b_{11}(t), ..., b_{NL}(t)\} \)

(25)

is a finite dimensional and closed subspace of the Hilbert space \( H = L^2[0,1] \). Therefore, \( Y \) is a complete subspace, and there is a unique best approximation out of \( Y \), such as \( f_{NM} \in Y \) for each \( f \in H \), that is \( \forall y \in Y, \| f - f_{NM} \| \leq \| f - y \| \). Since \( f_{NM} \in Y \), there exist unique coefficients \( c_{10}, c_{20}, ..., c_{NM} \), so that [31]

\[ f(t) \approx f_{NM}(t) = \sum_{n=1}^{N} \sum_{m=0}^{M} c_{nm} b_{nm}(t) = C^T B(t), \]  

(26)

where \( C \) and \( B(t) \) are the following vectors:

\[ B^T(t) = \begin{bmatrix} b_{10}(t), b_{20}(t), ..., b_{NO}(t), b_{11}(t), ..., b_{NL}(t) \end{bmatrix}, \]  

\[ C^T = \begin{bmatrix} c_{10}, c_{20}, ..., c_{NO}, c_{11}, ..., c_{NL} \end{bmatrix}. \]  

(27)

### III. LRLFI AND RRLFI OPERATORS FOR BERNOULLI HYBRID FUNCTIONS

In this section, LRLFI and RRLFI operators, are obtained for hybrid functions which, are shown with \( I^\alpha_L \) and \( I^\beta_R \), respectively. We divide the interval \([0, 1]\) into three subintervals including \( \left[0, \frac{n-1}{N}\right], \left[\frac{n-1}{N}, \frac{n}{N}\right], \text{ and } \left[\frac{n}{N}, 1\right] \)

according to the properties of the hybrid functions.

#### A. LRLFI Operator

In this part, we use the new method to obtain the LRLFI operator for hybrid functions. Suppose

\[ I^\alpha_L B(t) = I^\alpha_L b_{10}(t), I^\alpha_L b_{11}(t), ..., I^\alpha_L b_{NO}(t), I^\alpha_L b_{12}(t), ..., I^\alpha_L b_{NL}(t), \]

(28)

we obtain \( I^\alpha_L b_{nm}(t) \) for \( t \in \left[\frac{n-1}{N}, \frac{n}{N}\right], n = 1,2,...,N, m = 0,1,2,...,M \). So from Eqs. (4) and (21), and by applying the definition of the convolution, we have

\[ I^\alpha_L b_{nm}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} b_{nm}(\tau) d\tau \]  

(29)

\[ = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \beta_m(N(t-\tau) - n + 1) d\tau \]  

\[ = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \beta_m(N - n + 1) d\tau \]  

\[ = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \beta_m(NT - \tau) d\tau = d^\alpha_L \beta_m(NT). \]  

(30)

From the Eqs. (18) and (29), and by using the definition of Bernoulli polynomials (22), we get

\[ d^\alpha_L \beta_m(NT) = \sum_{k=0}^{m} \binom{m}{k} N^k \alpha_{m-k} d^\alpha_L (T^k) \]  

\[ = \sum_{k=0}^{m} \binom{m}{k} N^k \alpha_{m-k} \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} T^{k+\alpha}. \]  

(31)

Then, the Eq. (30) yields

\[ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \beta_m(N(t-\tau) - n + 1) d\tau \]  

\[ = \sum_{k=0}^{m} \binom{m}{k} N^k \alpha_{m-k} \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} (t - \frac{n-1}{N})^{k+\alpha}. \]  

(32)
In addition, for \( t \in \left[ \frac{n}{N}, 1 \right) \), by using the Equs. (24) and (28), we have

\[
I^\alpha_R b_{nm}(t) = \frac{1}{\Gamma(\alpha)} \int_{\frac{n}{N}}^{t} \tau^{\alpha-1} b_{nm}(t - \tau) d\tau
\]

\[
= \frac{1}{\Gamma(\alpha)} \int_{\frac{n}{N}}^{t} \tau^{\alpha-1} \beta_m(N(t - \tau) - n + 1) d\tau
\]

\[
= \frac{1}{\Gamma(\alpha)} \left( \int_{0}^{\frac{n}{N}} \tau^{\alpha-1} \beta_m(N(t - \tau) - n + 1) d\tau - \int_{0}^{t - \frac{n}{N}} \tau^{\alpha-1} \beta_m(N(t - \tau) - n + 1) d\tau \right).
\]  
(32)

The first integral in relation (32) is equal to Eqn. (31) and the second integral is calculated using the Eqn. (23) as follows:

\[
\frac{1}{\Gamma(\alpha)} \int_{0}^{\frac{n}{N}} \tau^{\alpha-1} \beta_m(N(t - \tau) - n + 1)
\]

\[
= \frac{1}{\Gamma(\alpha)} \int_{0}^{\frac{n}{N}} \tau^{\alpha-1} \beta_m(1 - (-N(t - \tau) + n)) d\tau
\]

\[
= \left( -1 \right)^m \frac{1}{\Gamma(\alpha)} \int_{0}^{\frac{n}{N}} \tau^{\alpha-1} \beta_m \left( -N \left( \frac{t - n}{N} \right) - t \right) d\tau,
\]  
(33)

by substituting \( Z = t - \frac{n}{N} \) into Eqn. (33), we have

\[
\left( -1 \right)^m \frac{1}{\Gamma(\alpha)} \int_{0}^{Z} \tau^{\alpha-1} \beta_m(-NZ + t) d\tau = \left( -1 \right)^m \sum_{k=0}^{m} \left( \begin{smallmatrix} m \\ k \end{smallmatrix} \right) (-1)^k \beta_m \left( \frac{N(k + 1)}{N} \right) \frac{1}{\Gamma(\alpha)} \left( t - \frac{n}{N} \right)^{k+\alpha}.
\]  
(34)

Following a similar approach to the one used in Eqn. (30), the Equs. (32) to (34) yield

\[
\frac{1}{\Gamma(\alpha)} \int_{0}^{\frac{n}{N}} \tau^{\alpha-1} \beta_m(N(t - \tau) - n + 1) d\tau
\]

\[
= \left( -1 \right)^m \sum_{k=0}^{m} \left( \begin{smallmatrix} m \\ k \end{smallmatrix} \right) (-1)^k \beta_{m-k} \left( \frac{N}{N} \right)^k \frac{1}{\Gamma(\alpha)} \left( t - \frac{n}{N} \right)^{k+\alpha}.
\]  
(35)

Finally, using the preceding results, we can write

\[
I^\alpha_R b_{nm}(t) = \begin{cases} 
0, & t \in \left[ 0, \frac{n-1}{N} \right), \\
R_{nm}(t), & t \in \left[ \frac{n-1}{N}, \frac{n}{N} \right), \\
R_{nm}(t) - S_{nm}(t), & t \in \left[ \frac{n}{N}, 1 \right), 
\end{cases}
\]  
(36)

where \( R_{nm}(t) \) and \( S_{nm}(t) \) are the relations (31) and (35), respectively.

\[\Box\]

**B. RRLFI Operator**

Now, we obtain the RRLFI operator for hybrid functions.

Suppose \( t \in \left[ \frac{n-1}{N}, \frac{n}{N} \right), n = 1, 2, \ldots, N \), and \( I^\alpha_R B(t) = I^\alpha_R B(t) \), from Eqn. (5) and Bernoulli hybrid functions (24), we have

\[
I^\alpha_R b_{nm}(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{\frac{n}{N}} \tau^{\alpha-1} b_{nm}(t - \tau) d\tau
\]

\[
= \frac{1}{\Gamma(\alpha)} \int_{0}^{\frac{n}{N}} \tau^{\alpha-1} \beta_m(Nt - n + 1) d\tau
\]

\[
= \frac{1}{\Gamma(\alpha)} \int_{0}^{\frac{n}{N}} \tau^{\alpha-1} \beta_m \left( \frac{N}{N} - t \right) d\tau.
\]  
(37)

For \( t \in \left[ 0, \frac{n-1}{N} \right) \), from Eqns. (5) and (24), the RRLFI operator is obtained as follows:

\[
I^\alpha_R b_{nm}(t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{1} \tau^{\alpha-1} b_{nm}(t) d\tau
\]

\[
= \frac{1}{\Gamma(\alpha)} \int_{t}^{\frac{n}{N}} \tau^{\alpha-1} b_{nm}(t) d\tau
\]

\[
= \frac{1}{\Gamma(\alpha)} \int_{t}^{\frac{n}{N}} \tau^{\alpha-1} b_{nm}(t) d\tau
\]

\[
= \frac{1}{\Gamma(\alpha)} \int_{t}^{\frac{n}{N}} \tau^{\alpha-1} b_{nm}(t) d\tau
\]

\[
= \frac{1}{\Gamma(\alpha)} \int_{t}^{\frac{n}{N}} \tau^{\alpha-1} b_{nm}(t) d\tau.
\]  
(38)

By applying the Eqns. (37) and (38), we deduce that

\[
I^\alpha_R b_{nm}(t) = \left( -1 \right)^m \sum_{k=0}^{m} \left( \begin{smallmatrix} m \\ k \end{smallmatrix} \right) (-1)^k \beta_{m-k} \left( \frac{N}{N} \right)^k \frac{1}{\Gamma(\alpha)} \left( t - \frac{n}{N} \right)^{k+\alpha}.
\]  
(39)
\[
\begin{aligned}
\dot{x}(t) &= A^T B(t), \\
\dot{\lambda}(t) &= C^T \bar{B}(t),
\end{aligned}
\]
(41)
(42)
where
\[
B^T(t) = [b_{10}(t), b_{20}(t), \ldots, b_{N1}(t), b_{21}(t),
\ldots, b_{nt}(t), b_{2m}(t), \ldots, b_{nm}(t)],
\]
(43)
\[
A^T = [a_{10}, a_{20}, \ldots, a_{N1}, a_{11}, \ldots, a_{21}, \ldots, a_{1m}, \ldots, a_{nm}],
\]
(44)
\[
\bar{B}^T(t) = [b_{10}(t), b_{20}(t), \ldots, b_{N1}(t), b_{21}(t),
\ldots, b_{nt}(t), b_{2m}(t), \ldots, b_{nm}(t)],
\]
(45)
\[
C^T = [c_{10}, c_{20}, \ldots, c_{N1}, c_{11}, \ldots, c_{21}, \ldots, c_{1m}, \ldots, c_{nm}].
\]
(46)

### IV. DESCRIPTION OF THE PROPOSED METHOD

In this section, we find \( x(t) \) and \( u(t) \) such that the constraint (2) and the initial condition (3) are satisfied and the objective function (1) of the problem becomes minimum. To this end, we transform the necessary optimality conditions of the FOCP into an equivalent fractional TPBVP. Then, using fractional operators and the collocation method, TPBVP is converted to a system of algebraic equations. By solving this system, the state variable \( x(t) \), the costate variable \( \lambda(t) \), and finally the control variable \( u(t) \), are determined.

#### A. Necessary Optimality Conditions

In this part, we obtain the necessary optimality conditions for FOCP in the form of fractional TPBVP according to the proposed method based on the LCFD and RCFD operators.

**Theorem 1** If \((x(t), u(t))\) be the optimal solution of FOCP expressed in Eqs. (1) to (3), there will be a costate function \( \dot{\lambda}(t) \) such that
\[
\begin{aligned}
K_2 \dot{\lambda}(t) + K_1 \dot{x}(t) + K_2 \dot{c}^\sigma D_t^\sigma \lambda(t) &= P(t, x(t), \lambda(t)), \\
\dot{x}(t) &= x_0, \quad \dot{\lambda}(t_f) = 0,
\end{aligned}
\]
(40)
where \( P \) and \( Q \) are the known functions in terms of \( x(t) \) and \( \lambda(t) \).

**Proof** This theorem is stated in reference [32] based on RRLFD. For compatibility with the proposed method, we replace RRLFD with RCFD. For this purpose, from Eq. (14), RRLFD of the costate function can be written in the following form:
\[
\dot{\lambda}(t) = \frac{\dot{c}^\sigma D_t^\sigma \lambda(t)}{1 - a}(t_f - t)^{-a},
\]
(47)
since \( \dot{\lambda}(t_f) = 0 \), we apply \( \dot{\lambda}(t) \) in the first Eq. (40) instead of the RRLFD. Also, using the necessary optimality conditions, the control variable can be represented in terms of costate and state variables. \( \square \)

#### B. Solving the Fractional TPBVP

In this part, the Bernoulli hybrid functions, LRLFI and RRLFI operators, and Legendre collocation method are used to approximate the solution of the mentioned fractional TPBVP. For solving the fractional differential equations (40) without loss of generality, let \( t_0 = 0 \) and \( t_f = 1 \). Then, we expand the first-order derivative of the state function \( \dot{x}(t) \) and costate function \( \dot{\lambda}(t) \) by the hybrid function as follows:

\[
\begin{aligned}
\dot{x}(t) &= A^T B(t), \\
\dot{\lambda}(t) &= C^T \bar{B}(t),
\end{aligned}
\]
(41)
(42)
where
\[
B^T(t) = [b_{10}(t), b_{20}(t), \ldots, b_{N1}(t), b_{21}(t),
\ldots, b_{nt}(t), b_{2m}(t), \ldots, b_{nm}(t)],
\]
(43)
\[
A^T = [a_{10}, a_{20}, \ldots, a_{N1}, a_{11}, \ldots, a_{21}, \ldots, a_{1m}, \ldots, a_{nm}],
\]
(44)
\[
\bar{B}^T(t) = [b_{10}(t), b_{20}(t), \ldots, b_{N1}(t), b_{21}(t),
\ldots, b_{nt}(t), b_{2m}(t), \ldots, b_{nm}(t)],
\]
(45)
\[
C^T = [c_{10}, c_{20}, \ldots, c_{N1}, c_{11}, \ldots, c_{21}, \ldots, c_{1m}, \ldots, c_{nm}].
\]
(46)

### V. ERROR BOUNDS

In this section, the error bounds for the hybrid functions approximation, LRLFI and RRLFI operators, and the proposed method are presented.
A. Error Bound for Hybrid Functions Approximation

In the following theorem, the upper bound for the error and the convergence of Bernoulli hybrid functions approximation are investigated.

**Theorem 2** Suppose that \( f(t) \in C^{(M+1)}[0,1] \) and \( f_{NM}(t) = \sum_{n=1}^{N} \sum_{m=0}^{M} c_{nm} b_{nm}(t) \), be the best approximation \( f(t) \) out of \( Y \) given in (25) and (26), the following inequality is satisfied:

\[
\| f(t) - f_{NM}(t) \|_2 \leq \frac{L}{(M+1)N(M+1)\sqrt{2M+3}} \tag{57}
\]

where

\[
L = \max |f^{(M+1)}(t)|, \quad t \in [0,1].
\]

**Proof** We divide interval \([0,1]\) into subintervals \([n-1\frac{n}{N}, n\frac{n}{N}], n = 1,2, \ldots, N\), with the limitation that \( f_n \) approximates \( f \) over the subinterval \( [n-1\frac{n}{N}, n\frac{n}{N}], n = 1,2, \ldots, N \) and \( f(t) \approx \sum_{n=1}^{N} f_n(n) \). Let the Taylor polynomials

\[
f_n(t) = \sum_{k=0}^{N-1} f(n) \left( t - n \frac{k}{N} \right) \frac{(n-1)}{k!},
\]

we know that

\[
|f(t) - f_n(t)| \leq |f^{(M+1)}(\xi)\left( t - n \frac{k}{N} \right)(n-1)\frac{(n-1)}{(M+1)!} | dt
\]

Assume that \( f_{NM}(t) = \sum_{n=1}^{M} \sum_{m=0}^{N} c_{nm} b_{nm}(t) \), from Eqs. (24), (58), and (59), we have

\[
\| f(t) - f_{NM}(t) \|_2 \leq \sum_{n=1}^{N} \int_{0}^{1} |f(t) - f_{NM}(t)|^2 dt
\]

\[
\leq \sum_{n=1}^{N} \int_{0}^{1} |f(t) - f_{NM}(t)|^2 dt
\]

\[
\leq \sum_{n=1}^{N} \int_{0}^{1} f^{(M+1)}(\xi) \left( t - n \frac{k}{N} \right)^2 \frac{(n-1)^2}{(M+1)!} | dt
\]

\[
\leq \frac{L^2}{(2M+3)((M+1)!)^2N^2}\tag{60}
\]

by taking square roots, the proof is finished. This theorem shows that the Bernoulli hybrid functions approximation error tends to zero if \( M \) and \( N \) are sufficiently increased. This result confirms that \( f_{NM} \) converges to \( f \).

**B. Error Bound for the LRLFI Operator**

**Theorem 3** Suppose \( f(t) \in C^{(M+1)}[0,1] \) and \( 0 < \alpha \leq 1 \), the error bound for the LRLFI operator is achieved as follows:

\[
\| L_{\alpha}^2 f(t) - L_{\alpha}^2 f_{NM}(t) \|_2 \leq \frac{L}{(\alpha+1)(M+1)\sqrt{2M+3}} \tag{61}
\]

**Proof** This theorem is proved by using inequalities (19) and (57).

**C. Error Bound for the RRLFI Operator**

**Theorem 4** Assume \( f(t) \in C^{(M+1)}[0,1] \). For \( 0 < \alpha \leq 1 \), the error bound for the RRLFI operator is defined by:

\[
\| L_{\alpha}^2 f(t) - L_{\alpha}^2 f_{NM}(t) \|_2 \leq \frac{L}{(\alpha+1)(M+1)\sqrt{2M+3}}
\]

**D. Error Bound for the Proposed Method**

In this section, we estimate the error of the proposed method with respect to the hybrid functions order \( N, M, \tilde{N}, \tilde{M} \). This theorem shows while the dimensions of the basis functions are increased, the error bounds tend to zero, consequently the state and control approximate variables converge to the exact values.

**Theorem 5** Suppose \( x(t) \) and \( \lambda(t) \) are the exact solutions of the TPBV (40), \( x(t) \) and \( \lambda(t) \in C^{(M+2)}[0,1] \), \( x_{NM}(t) = \lambda_{NM}(t) \in C^{(M+2)}[0,1] \), the approximate solutions of \( x(t) \) and \( \lambda(t) \) where achieved from Eqs. (47) and (49). Also \( P(t, x(t), \lambda(t)) \) and \( Q(t, x(t), \lambda(t)) \) are Lipschitz functions, with the Lipschitz constants \( P_0, Q_0 \), for \( i = 1, 2 \). The error bounds of (40) showing with \( E_1 \) and \( E_2 \), for the proposed method are obtained as follows:

\[
\| E_1 \|_2 \leq \frac{P L_1}{(M+1)\sqrt{2M+3}} + \frac{K_1 P L_2 Q_2}{(M+1)\sqrt{2M+3}},
\]

\[
\| E_2 \|_2 \leq \frac{P L_1}{(M+1)\sqrt{2M+3}} + \frac{K_1 P L_2 Q_2}{(M+1)\sqrt{2M+3}}.
\]

where \( L_1 = \max |x^{(M+2)}(t)|, t \in [0,1] \) and \( L_2 = \max |\lambda^{(M+2)}(t)|, t \in [0,1] \).
In this section, some examples are presented to illustrate the efficiency and accuracy of the proposed method. The obtained results have been compared with those reported by using other methods.

**Example 1** Consider the following two-dimensional FOCP [33]:

$$\min J = \int_0^1 \left( (x_1(t) - 1 - t^{1.5})^2 + (x_2(t) - t^{2.5})^2 + \left( u(t) - \frac{3\sqrt{\pi}}{4}t + t^{2.5} \right)^2 \right) dt,$$

subject to:

$$\frac{\partial D_1^{0.5}x_1(t)}{\partial t} = x_2(t) + u(t),$$

$$\frac{\partial D_1^{0.5}x_2(t)}{\partial t} = x_1(t) + \frac{15\sqrt{\pi}}{16}t^2 - t^{1.5} - 1,$$

$$x_1(0) = 1, \quad x_2(0) = 0.$$

For this problem, $x_1(t) = 1 + t^{1.5}, \quad x_2(t) = t^{2.5},$ and $u(t) = \frac{3\sqrt{\pi}}{4}t - t^{2.5}$ minimize the cost function and the minimum value is $J = 0$. The necessary optimality conditions are as follows:

$$\frac{\partial D_1^{0.5}x_1(t)}{\partial t} = x_2(t) + \frac{3\sqrt{\pi}}{4}t - t^{2.5} - \frac{1}{2}A_1(t),$$

$$\frac{\partial D_1^{0.5}x_2(t)}{\partial t} = x_1(t) + \frac{15\sqrt{\pi}}{16}t^2 - t^{1.5} - 1,$$

$$\frac{\partial D_1^{0.5}A_1(t)}{\partial t} = A_2(t) - 2t^{1.5} + 2x_1(t) - 2, \quad (64)$$

$$\frac{\partial D_1^{0.5}A_2(t)}{\partial t} = A_1(t) - 2t^{2.5} + 2x_2(t),$$

$$u(t) = \frac{1}{2}A_1(t) - \frac{3\sqrt{\pi}}{4}t + t^{2.5} = 0,$$

$$x_1(0) = 1, \quad x_2(0) = 0, \quad A_1(1) = 0, \quad A_2(1) = 0.$$

By applying the proposed approach and Equations (53) and (54), for $N = 1, M = 2$ and $M = 1, M = 1$, we obtain

$$x_1(t) = A_1^{1.5}B(t) = a_{10}^{1.5}b_{10}(t) + a_{11}^{1.5}b_{11}(t) + a_{12}^{1.5}b_{12}(t),$$

$$x_2(t) = A_2^{1.5}B(t) = c_{01}^{1.5}b_{01}(t) + c_{11}^{1.5}b_{11}(t) + c_{12}^{1.5}b_{12}(t),$$

$$A_1(t) = d_{10}^{1.5}b_{10}(t) + d_{11}^{1.5}b_{11}(t),$$

$$A_2(t) = e_{01}^{1.5}b_{01}(t) + e_{11}^{1.5}b_{11}(t).$$

Using the Equations (55) and (56) yield

$$x_1(t) = A_1^{1.5}B(t) + x_1(0),$$

$$x_2(t) = A_2^{1.5}B(t) + x_2(0).$$

**Example 2** We consider the following FOCP with variable fractional order [35]:

$$\min J = \int_0^1 \left( \frac{1}{2}x(t)^2 + \frac{1}{2}x(t) \right) dt,$$

subject to the dynamical system

$$x(t) + \frac{\partial D_f^{\alpha}x(t)}{\partial t} = u(t) + t^2, \quad x(0) = 0, \quad x(1) = \frac{2}{\Gamma(\alpha+3)}.$$
For this problem, the exact solution is \((x(t), u(t)) = \left(\frac{2t^{(a+2)}}{\Gamma(a+3)} + \frac{2t^{(a+1)}}{\Gamma(a+2)}\right)^2\). The necessary optimality conditions are as follows:
\[
2t\left(\frac{\partial^2 x}{\partial t^2} - (a + 2)x(t)\right) + \lambda(t) = 0, \tag{67}
\]

\[
\dot{\lambda}(t) - \frac{\partial^2 \lambda}{\partial t^2} = -\lambda(t) \frac{(a+2)}{t},
\]
\[
\dot{x}(t) + \frac{\partial^2 x}{\partial t^2} = u(t) + t^2, \quad x(0) = 0, \lambda(1) = 0.
\]
From Eq. (67), we have
\[
u(t) = \frac{(a+2)}{2t^2} x(t) - \frac{\lambda(t)}{2t^2} \tag{68}
\]
Here, we approximate the unknown functions by the hybrid functions as follows:
\[
\dot{\lambda}(t) = d_{10}b_{10}(t) + d_{11}b_{11}(t),
\]
\[
\lambda(t) = -d_{10}l_1^{1}\bar{b}_{10}(t) - d_{11}l_1^{1}\bar{b}_{11}(t),
\]
\[
\frac{\partial^2 \lambda}{\partial t^2} = -d_{10}l_1^{1-\alpha}\bar{b}_{10}(t) - d_{11}l_1^{1-\alpha}\bar{b}_{11}(t).
\]
\[
\dot{x}(t) = A^T B(t), \quad x(t) = A^T l_1^{1} B(t),
\]
\[
\frac{\partial^2 x}{\partial t^2} = A^T l_1^{1-\alpha} B(t).
\]

We replaced Equ. (68) and (69) in necessary optimality conditions (67) and solved the resulting equations. This problem was solved in [32] by two algorithms (Alg.1 and Alg.2). The Alg.1 is based on calculating the necessary optimality conditions and solves the resulted equations using the spectral method. In Alg.2 the state function was first discretized using the numerical integration, followed by the Rayleigh-Ritz method to evaluate state and control functions.

A direct approach based on Chebyshev polynomials and the Legendre-Gauss quadrature formula is employed to solve this FOCP [35]. The comparison of maximum absolute errors in the state and control variables of the present method with those of proposed numerical schemes in [32] and [35] is shown in Tables II and III. Figs. 3 and 4 display the absolute errors of \(x(t)\) and \(u(t)\) by selecting the different values of \(N\) and \(M\) at \(\alpha = 0.5\). These graphs show that the error of the solutions is decreased by increasing the number of basis functions. The exact and approximate solutions of state and control variables for different values of \(\alpha\) are depicted in Fig. 5.

**Example 3** As a practical and nonlinear example, consider the optimal maneuvers of a rigid asymmetric spacecraft. The Euler equations for the angular velocities \(x_1(\tau), x_2(\tau), \) and \(x_3(\tau)\)
of spacecraft are given by

$$
x_1(t) = -\frac{h_2}{l_1} x_2(t)x_3(t) + u_1(t)
$$

$$
x_2(t) = -\frac{h_2}{l_2} x_1(t)x_3(t) + u_2(t)
$$

$$
x_3(t) = -\frac{h_1}{l_3} x_1(t)x_2(t) + u_3(t)
$$

where $u_1$, $u_2$, and $u_3$ are control the torques. The spacecraft principle inertia are $l_1 = 86.24 \text{ Kg m}^2$, $l_2 = 85.07 \text{ Kg m}^2$, and $l_3 = 113.59 \text{ Kg m}^2$. Also, $\frac{h_2}{l_1}$, $\frac{h_2}{l_2}$, and $\frac{h_1}{l_3}$ are the inertia difference ratios. The performance index to be minimized is expressed by

$$
\min \int_0^t \left( u_1^2(t) + u_2^2(t) + u_3^2(t) \right) dt,
$$

The boundary conditions are as follows:

$$
x_1(0) = 0.01 \quad x_2(0) = 0.005 \quad x_3(0) = 0.001
$$

According to the proposed method, the following TPBVP should be solved:

$$
\dot{x}_1(t) = -\frac{h_2}{l_1} x_2(t)x_3(t) - \frac{h_1}{l_1} x_1(t)
$$

$$
\dot{x}_2(t) = -\frac{h_2}{l_2} x_1(t)x_3(t) - \frac{h_2}{l_2} x_2(t)
$$

$$
\dot{x}_3(t) = -\frac{h_1}{l_3} x_1(t)x_2(t) - \frac{h_1}{l_3} x_3(t)
$$

$$
\dot{\lambda}_1(t) = \frac{h_2}{l_1} x_3(t)\lambda_2(t) + \frac{h_1}{l_1} x_1(t)\lambda_3(t)
$$

$$
\dot{\lambda}_2(t) = \frac{h_2}{l_2} x_3(t)\lambda_1(t) + \frac{h_2}{l_2} x_2(t)\lambda_3(t)
$$

$$
\dot{\lambda}_3(t) = \frac{h_1}{l_3} x_2(t)\lambda_1(t) + \frac{h_1}{l_3} x_1(t)\lambda_2(t)
$$

Also, the optimal control variables are obtained as follows:

$$
u_1(t) = -\frac{x_1(t)}{l_1}, \quad u_2(t) = -\frac{x_2(t)}{l_2}, \quad and \quad u_3(t) = -\frac{x_3(t)}{l_3}
$$

We use transformation $\tau = 100t, t \in [0, 1]$ to apply the proposed method. By using this method with $N = 1, M = 7$, we get:

$$
x_1(t) = -2.468260116060948 \times 10^{-16} t^8
$$

$$-3.823003555219281 \times 10^{-12} t^7
$$

$$+2.722028451181812 \times 10^{-13} t^6 + 2.386478651140519 \times 10^{-10} t^5 - 1.195968612517397 \times 10^{-9} t^4
$$

$$-8.248480363939749 \times 10^{-7} t^3 + 2.479330170240148 \times 10^{-6} t^2 - 0.010001653525046 t + 0.01,
$$

$$x_2(t) = 3.957240843616689 \times 10^{-6} t^6
$$

$$-8.933546057097384 \times 10^{-15} t^7 + 5.561002887595278 \times 10^{-14} t^6 - 1.40658941719365 \times 10^{-13} t^5
$$

$$+7.027484806347674 \times 10^{-10} t^4 + 1.606375494884975 \times 10^{-6} t^3 - 4.821937043751208 \times 10^{-6} t^2
$$

$$-0.004996785005088 t + 0.005,
$$

$$x_3(t) = 1.707020832761955 \times 10^{-12} t^8
$$

$$-6.071238553944105 \times 10^{-12} t^7 + 7.682749774825881 \times 10^{-12} t^6 - 3.52934339794504 \times 10^{-11} t^5
$$

$$+1.550767918419068 \times 10^{-10} t^4 + 2.572588863294985 \times 10^{-7} t^3 - 7.7239218234617 \times 10^{-7} t^2
$$

$$-9.9948498910374 \times 10^{-4} t + 0.001.
$$

By substituting $t = 0$ and $t = 1$, boundary conditions are obtained. In Table IV, a comparison is created among the numerical results of the cost function $J$, generated by the
proposed hybrid functions method by taking $N = 1$, $M = 7$ with the reported results in [36] by applying fractional order Chebyshev functions, [37] by using a quasilinearization technique based on the Chebyshev polynomials, and [38] by adopting Fibonacci wavelets and the Galerkin method. These results demonstrate the accuracy and efficiency of the proposed approach in comparison with mentioned methods. State and control approximate variables are shown in Figs. 6 and 7.

Fig. 6. The numerical values of state variables for Example 3.

Fig. 7. The estimate values of control variables in Example 3.
TABLE II
THE MAXIMUM ERROR OF \( x(t) \) AT \( \alpha = 0.5 \) AND COMPARISON WITH OTHER METHODS FOR EXAMPLE 2

<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>( N = 2 )</td>
<td>3.10 \times 10^{-2}</td>
<td>2.73 \times 10^{-2}</td>
<td>2.16 \times 10^{-2}</td>
<td>1.02 \times 10^{-3} (N = 2, M = 2)</td>
</tr>
<tr>
<td>( N = 5 )</td>
<td>3.55 \times 10^{-4}</td>
<td>1.60 \times 10^{-4}</td>
<td>1.72 \times 10^{-4}</td>
<td>3.26 \times 10^{-5} (N = 1, M = 5)</td>
</tr>
</tbody>
</table>

TABLE III
THE MAXIMUM ERROR OF \( u(t) \) AT \( \alpha = 0.5 \) AND COMPARISON WITH OTHER METHODS FOR EXAMPLE 2

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( N = 2 )</td>
<td>2.04 \times 10^{-1}</td>
<td>2.56 \times 10^{-1}</td>
<td>2.44 \times 10^{-1}</td>
<td>6.18 \times 10^{-2} (N = 2, M = 2)</td>
</tr>
<tr>
<td>( N = 5 )</td>
<td>9.13 \times 10^{-3}</td>
<td>8.22 \times 10^{-3}</td>
<td>6.26 \times 10^{-3}</td>
<td>3.44 \times 10^{-3} (N = 1, M = 5)</td>
</tr>
</tbody>
</table>

TABLE IV
COMPARISON OF THE OPTIMAL VALUE \( J \) WITH METHODS IN [36], [37], AND [38] FOR EXAMPLE 3

<table>
<thead>
<tr>
<th>Method</th>
<th>Optimal cost function ( J )</th>
<th>Approximation order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fractional Chebyshev functions [36]</td>
<td>0.004687795</td>
<td>( m = 10 )</td>
</tr>
<tr>
<td>Quasilinearization [37]</td>
<td>0.00534063</td>
<td>( N = 10 )</td>
</tr>
<tr>
<td>Fibonacci wavelets and Galerkin [38]</td>
<td>0.005762040</td>
<td>( k = 2, M = 4 )</td>
</tr>
<tr>
<td>Present method</td>
<td>0.004687795353</td>
<td>( N = 1, M = 7 )</td>
</tr>
</tbody>
</table>

VII. CONCLUSION

In this paper, the Bernoulli hybrid functions indirect method is presented to solve integer-fractional OCPs. Here, the LRLFI and RRLFI operators are computed for mentioned hybrid functions directly and without any approximation. By determining the necessary optimality conditions, the solution of the FOCP is transformed into solving TPBVP, including a system of FDEs. Then, the resulted system is solved, using the hybrid functions approximation and LRLFI and RRLFI operators as well as the collocation method. The error bounds and convergence of the proposed method are investigated. Finally, the method is illustrated by some test problems. The obtained numerical results are compared with the exact solutions and some of the ones available to display the accuracy and proficiency of the proposed method. As can be seen, with a few numbers of the hybrid basis functions, satisfactory results are obtained.

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Fractional control, matrix of integration,


Seyed Mehdi Shafiof was born in 1980 in Isfahan, Iran. He received the B.S. degree in Applied Mathematics from K.N. Toosi University of Technology (KNTU), Tehran, Iran, in 2004 and M.S. degree in Applied Mathematics from the Guilan University, Rasht, Iran, in 2006, and the Ph.D. degree in Control and Optimization from the Payame Noor University, Tehran, Iran, in 2020. In 2008, he joined the Payame Noor University as an Instructor in the Department of Mathematics. His current research interests include fractional optimal control problems and fractional differential equations.

Javad Askari-Marnani was born in 1964 in Isfahan, Iran. He received the B.Sc. and M.Sc. degrees in electrical engineering from the Isfahan University of Technology in 1987 and the University of Tehran in 1993, respectively. He received also Ph.D. degree in electrical engineering from the University of Tehran in 2001 and under the supervision of Professor Jabedar. From 1988 to 1990 he worked at Isfahan Petrochemical Company in Isfahan. From 1999 to 2001, he received a grant from DAAD and joined the Control Engineering department at Technical University Hamburg, Hamburg, Germany, where he completes his Ph.D. with Professor Lunze’s research group. He is currently an associate professor at the control engineering department of the Isfahan University of Technology. His current research interests are in control theory, particularly in the field of fault diagnosis and fault-tolerant control, adaptive control of time-delay systems, multi-agent systems, identification, and electrical engineering curriculum.

Maryam Shams Solary was born in Tehran, Iran, in 1978. She received the B.S. degree from the Faculty of Mathematics, Isfahan University of Technology, Isfahan, Iran, in 2000. She received her M.S. and Ph.D. degrees in Applied Mathematics from the Guilan University, Iran, in 2002 and 2009, respectively. In 2010, she joined the Payame Noor University as an Assistant Professor in the Department of Mathematics. From September 2008 to April 2009, she was a Visiting Student at the Hamburg University of Technology, Hamburg, Germany that she worked by Siegfried M. Rump (Head of the Institute for Reliable Computing, Hamburg University of Technology). Her current research interests include Applied Mathematics, Numerical Analysis, Numerical Linear Algebra, Matrix Theory.