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Hybrid Functions to Solve Fractional Optimal Control Problems Using the Collocation Method

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ABSTRACT

This article aims to introduce a modern numerical method based on the hybrid functions, consisting of the Bernoulli polynomials and Block-Pulse functions. An indirect approach is proposed for solving the fractional optimal control problems (FOCPs). Firstly, the two-point boundary value problem (TPBVP) is calculated for a class of FOCPs, including integer-fractional derivatives, leading to a system of fractional differential equations (FDEs), which have the left and right-sided Caputo fractional derivatives (CFD). Therefore, a new approach is proposing to achieve the left Riemann-Liouville fractional integral (LRLFI) and right Riemann-Liouville fractional integral (RRLFI) operators for Bernoulli hybrid functions. Then, hybrid functions approximation, LRLFI and RRLFI operators, and the collocation method are used to solve the TPBVP. The error bounds for the hybrid function and LRLFI and RRLFI operators are also presented. Moreover, the convergence of the proposed method is proved. Finally, the simplicity and accuracy of the method are illustrated using some numerical examples.

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I. INTRODUCTION

In 1695, the idea of fractional calculus was firstly discussed in some correspondence between Leibniz and Hopital. Leibniz predicted that "one day, the useful consequences would be drawn," although his vision has been realized during the following years. Later, fractional calculus became a fascinating area for mathematicians, and many forms of fractional differential operators were proposed. The most popular ones are the Riemann-Liouville and Caputo fractional derivatives [1]-[4]. Fractional calculus is an effective modeling tool that displays the behavior of many mechanical and biological dynamics better than the integer models [5]. Recently, fractional calculus has been widely used in different fields of applied sciences such as physics, engineering, and bioengineering [6], stochastic systems [7], and the theory of diffusion [8]. One of the latest applications of fractional models is to investigate and analyze the complex transmission pattern of COVID-19 disease [9], [10].

The optimal control problems have been studied in almost many fields to provide broad background knowledge. These problems minimize an objective function, subject to dynamic constraints on the state and control variables [11], [12]. FOCPs are a subclass of optimal control problems, which objective function or constraint equation governed by FDE [13]. Since the dynamic constraints of these problems involve FDE, finding the exact solution is very complicated, so numerical methods must be utilized to solve FOCPs. These numerical methods can be divided into two major classes, including direct and indirect methods. The indirect methods are based on Pontryagin's maximum principle and the necessary optimality conditions for FOCPs. These conditions lead to a TPBVP that can analytically or numerically be solved using widely known methods for differential equations. For this purpose, Agrawal used the calculus of variation to specify the optimality conditions for FOCPs, based on the Riemann-Liouville fractional derivatives (RLFD) [14]. He also set the optimality conditions for FOCPs containing CFD [15]. The TPBVP,

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including left and right fractional derivatives, have approximately been solved by the Legendre multiwavelet collocation method [16], variational iteration and Adomian decomposition methods [17], and fractional power series neural network method [18]. The advantages of indirect methods are explained in [19], [20]. In the direct methods, FOCP is solved by minimizing the objective function, using the approximation or discretization of unknown functions, according to the problem constraints, and without deriving the Hamiltonian equations. Some examples of this method are as follows: Hybrid of Block-Pulse functions and orthonormal Taylor polynomials [21], Bernoulli polynomials with the operational matrix of fractional integration [22], Genocchi operational matrix of integration [23], Fractional-order Bessel wavelet functions [24].

In recent years, many authors have applied hybrid functions to solve different problems because hybrid functions are a mathematical power tool to approximate the functions defined on the distinct subintervals. Hybrid functions have consisted of the combination of a polynomial with piecewise constant basis functions such as Block-Pulse. For more details about this topic, can refer to [25]-[27].

In the current article, we focus on the FOCP characterized by integer-fractional dynamical system as follows:

$$\min J = \int_{t_0}^{t_f} f\big(t, x(t), u(t)\big)dt, \tag{1}$$

subject to

$$K_1\dot{x}(t) + K_2\, {}_{t_0}^{C}D_t^{\alpha}x(t) = g\big(t, x(t)\big) + b(t)u(t), \tag{2}$$

with initial condition

$$x(t_0) = x_0, \qquad 0 < \alpha < 1, \tag{3}$$

where $f\big(t, x(t), u(t)\big)$, $g\big(t, x(t)\big)$ and $b(t)$ are smooth, $b(t)$ is non-zero and, K_1 and K_2 are scalar numbers. As a practical example, the dynamical system of light amplification in Erbium-doped fiber amplifier, one of the most commonly applied types of fiber amplifiers in metro optical networks, contains the integer-fractional derivatives [28].

This article presents a new method to obtain the LRLFI, and RRLFI operators for the hybrid of Bernoulli polynomials and Block-Pulse functions directly and without any approximation. The accuracy of these operators is effective in increasing the accuracy of the proposed method for solving FOCPs. We employ the following steps to solve the stated FOCP using the suggested indirect method.

First, TPBVP based on the left and right CFD is obtained from the necessary optimality conditions of the FOCP. Then, in the obtained FDEs, the highest order derivatives of the state $x(t)$ and costate $\lambda(t)$ functions are approximated by using the Bernoulli hybrid functions with unknown coefficients. Afterward, by applying the LRLFI, RRLFI operators, and Legendre-Gauss collocation method, the problem is converted to a system of algebraic equations, by the solution of which the unknown coefficients are obtained. Besides, estimation errors and convergence analysis are brought. The existing results demonstrate that by increasing the number of hybrid functions basis, the approximate solutions converge to the exact solutions.

The organization of this paper is as follows: In Section II, we provide some necessary definitions and properties of fractional calculus and hybrid functions. Section III, introduces LRLFI and RRLFI operators for hybrid functions. In Section IV, we obtain the necessary optimality conditions and describe the numerical method for solving FOCPs. We provide error bound and convergence analysis for the proposed method in Section V. In Section VI, test problems are used to verify the accuracy of the suggested approach. The final section of this article is the conclusion.

II. PRELIMINARIES AND NOTATIONS

We present some basic definitions and some required properties of the fractional calculus and hybrid functions used in this article.

A. *Fractional Calculus*

In this part, we briefly provide some definitions of the fractional derivative and integral, and their properties [1]-[4], [29]. Let $f:[a,b]\to R$ be a function, and $\alpha > 0$ be the order of fractional derivative and integral, and $m = [\alpha] + 1$.

Definition 1 The LRLFI and RRLFI operators of order α are respectively defined as follows:

$${}_aI_t^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \tag{4}$$

$${}_tI_b^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1} f(\tau)d\tau, \tag{5}$$

where $\Gamma(\alpha)$ is the Gamma function.

Definition 2 The left and right RLFD operators of order α are given by:

$${}_a^{RL}D_t^{\alpha}f(t) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_a^t (t-\tau)^{m-\alpha-1} f(\tau)\, d\tau, \tag{6}$$

$${}_t^{RL}D_b^{\alpha}f(t) = \frac{(-1)^m}{\Gamma(m-\alpha)}\frac{d^m}{dt^m}\int_t^b (\tau-t)^{m-\alpha-1} f(\tau)\, d\tau. \tag{7}$$

Definition 3 The left and right CFD operators are defined by:

$${}_a^{C}D_t^{\alpha}f(t) = \frac{1}{\Gamma(m-\alpha)}\int_a^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)d\tau, \tag{8}$$

$${}_t^{C}D_b^{\alpha}f(t) = \frac{(-1)^m}{\Gamma(m-\alpha)}\int_t^b (\tau-t)^{m-\alpha-1} f^{(m)}(\tau)d\tau. \tag{9}$$

Some properties of the Caputo derivative are as follows:

$${}_a^{C}D_t^{\alpha}\ {}_aI_t^{\alpha}f(t) = f(t),\ {}_t^{C}D_b^{\beta}\ {}_tI_b^{\beta}f(t) = f(t),\ 0 < \beta < 1. \tag{10}$$

$${}_aI_t^{\alpha}\,{}_a^{C}D_t^{\alpha}f(t) = f(t) - \sum_{i=0}^{m-1} f^{(i)}(a)\frac{(t-a)^i}{i!}. \tag{11}$$

$${}_a^{C}D_t^{\alpha}K = 0,\ (K \text{ is constant.}) \tag{12}$$

$$\begin{cases} {}_a^{C}D_t^{m}f(t) = f^{(m)}(t), \\ {}_t^{C}D_b^{m}f(t) = (-1)^m f^{(m)}(t), \quad m \in N, \\ {}_a^{C}D_t^{\alpha}f(t) = {}_aI_t^{m-\alpha}D^m f(t), \\ {}_t^{C}D_b^{\alpha}f(t) = (-1)^m\ {}_tI_b^{m-\alpha}D^m f(t), \qquad \alpha \notin N_0. \end{cases} \tag{13}$$

The relation between RCFD and RRLFD is as follows:

$$ {}_t^C D_b^\alpha f(t) = {}^{RL}_t D_b^\alpha f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{\Gamma(k+1-\alpha)} (b-t)^{k-\alpha}. \tag{14} $$

The following relations are established:

$$ {}_a I_t^\alpha \; {}_a I_t^\beta f(t) = {}_a I_t^{\alpha+\beta} f(t), \; {}_t I_b^\alpha \; {}_t I_b^\beta f(t) = {}_t I_b^{\alpha+\beta} f(t), \alpha, \beta > 0 \tag{15} $$

$$ {}_t I_b^\alpha \; {}_t^C D_b^\alpha f(t) = f(t) - \sum_{i=0}^{m-1} (-1)^i f^{(i)}(b) \frac{(b-t)^i}{i!}. \tag{16} $$

$$ {}_t I_b^\alpha (b-t)^{B-1} = \frac{\Gamma(B)}{\Gamma(B+\alpha)} (b-t)^{B+\alpha-1}, \quad \alpha, B > 0. \tag{17} $$

$$ {}_0 I_t^\alpha t^r = \frac{\Gamma(r+1)}{\Gamma(r+1+\alpha)} t^{r+\alpha}, \quad r \in N_0, \quad t > 0. \tag{18} $$

$$ \| {}_a I_t^\alpha f(t) \|_{L^2(a,b)} \le \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \| f(t) \|_{L^2(a,b)}, \quad 0 < \alpha \le 1. \tag{19} $$

$$ \| {}_t I_b^\alpha f(t) \|_{L^2(a,b)} \le \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \| f(t) \|_{L^2(a,b)}, \quad 0 < \alpha \le 1. \tag{20} $$

LRLFI operator satisfies the following formula:

$$ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau = \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * f(t), \tag{21} $$

where $t^{\alpha-1} * f(t)$ is the convolution of $t^{\alpha-1}$ and $f(t)$. Also, RLFI and CFD operators are linear operations.

B. Bernoulli Polynomials

Bernoulli polynomials of order m can be defined as [30]:

$$ \beta_m(t) = \sum_{k=0}^{m} \binom{m}{k} \alpha_{m-k} t^k, \tag{22} $$

where $\alpha_k = \beta_k(0), k = 0,1,2,\dots,m,$ are Bernoulli numbers. The Bernoulli polynomials satisfy the following formula:

$$ \beta_m(1-t) = (-1)^m \beta_m(t) \tag{23} $$

C. Hybrid of Block-Pulse Functions and Bernoulli Polynomials

Hybrid functions $b_{nm}(t)$ for $n = 1,2,\dots,N, m = 0,1,2,\dots,M$ are given over the interval $[0,1)$ as:

$$ b_{nm}(t) = \begin{cases} \beta_m(Nt-n+1), & t \in \left[\frac{n-1}{N}, \frac{n}{N}\right), \\ 0, & otherwise, \end{cases} \tag{24} $$

where n and m are the orders of the Block-Pulse functions and Bernoulli polynomials, respectively. It is clear that

$$ Y = span\{b_{10}(t), b_{20}(t), \dots, b_{N0}(t), b_{11}(t), \dots, b_{N1}(t), \dots, b_{1M}(t), \dots, b_{NM}(t)\}, \tag{25} $$

is a finite dimensional and closed subspace of the Hilbert space $H = L^2[0,1]$. Therefore, Y is a complete subspace, and there is a unique best approximation out of Y, such as $f_{NM} \in Y$ for each $f \in H$, that is $\forall y \in Y, \|f - f_{NM}\| \le \|f - y\|$. Since $f_{NM} \in Y$, there exist unique coefficients $c_{10}, c_{20}, \dots, c_{NM}$, so that [31]

$$ f(t) \simeq f_{NM}(t) = \sum_{n=1}^{N} \sum_{m=0}^{M} c_{nm} b_{nm}(t) = C^T B(t), \tag{26} $$

where C and $B(t)$ are the following vectors:

$$ B^T(t) = [b_{10}(t), b_{20}(t), \dots, b_{N0}(t), b_{11}(t), \dots, b_{N1}(t), \dots, b_{1M}(t), \dots, b_{NM}(t)], $$
$$ C^T = [c_{10}, c_{20}, \dots, c_{N0}, c_{11}, \dots, c_{N1}, \dots, c_{1M}, \dots, c_{NM}]. \tag{27} $$

III. LRLFI AND RRLFI OPERATORS FOR BERNOULLI HYBRID FUNCTIONS

In this section, LRLFI and RRLFI operators, are obtained for hybrid functions which, are shown with I_L^α and I_R^α, respectively. We divide the interval $[0,1]$ into three subintervals including $\left[0, \frac{n-1}{N}\right), \left[\frac{n-1}{N}, \frac{n}{N}\right)$ and $\left[\frac{n}{N}, 1\right)$ according to the properties of the hybrid functions.

A. LRLFI Operator

In this part, we use the new method to obtain the LRLFI operator for hybrid functions. Suppose ${}_0 I_t^\alpha B(t) = I_L^\alpha B(t)$, where

$$ I_L^\alpha B(t) = [I_L^\alpha b_{10}(t), \dots, I_L^\alpha b_{N0}(t), I_L^\alpha b_{11}(t), \dots, I_L^\alpha b_{N1}(t), \dots, I_L^\alpha b_{1M}(t), \dots, I_L^\alpha b_{NM}(t)]^T. $$

We obtain $I_L^\alpha b_{nm}(t)$ for $t \in \left[\frac{n-1}{N}, \frac{n}{N}\right), n = 1,2,\dots,N, m = 0,1,2,\dots,M$. So from Equs. (4) and (21), and by applying the definition of the convolution, we have

$$ I_L^\alpha b_{nm}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} b_{nm}(\tau) d\tau = \frac{1}{\Gamma(\alpha)} \int_0^t \tau^{\alpha-1} b_{nm}(t-\tau) d\tau. \tag{28} $$

By using the Equs. (24) and (28), and introducing the change of variable $T = t - \frac{n-1}{N}$, we have

$$ I_L^\alpha b_{nm}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \tau^{\alpha-1} b_{nm}(t-\tau) d\tau $$
$$ = \frac{1}{\Gamma(\alpha)} \int_0^{t-\frac{n-1}{N}} \tau^{\alpha-1} \beta_m(N(t-\tau) - n + 1) d\tau $$
$$ = \frac{1}{\Gamma(\alpha)} \int_0^{t-\frac{n-1}{N}} \tau^{\alpha-1} \beta_m\left(N\left(\left(t - \frac{n-1}{N}\right) - \tau\right)\right) d\tau $$
$$ = \frac{1}{\Gamma(\alpha)} \int_0^T \tau^{\alpha-1} \beta_m(N(T-\tau)) d\tau = {}_0 I_T^\alpha \beta_m(NT). \tag{29} $$

From the Equs. (18) and (29), and by using the definition of Bernoulli polynomials (22), we get

$$ {}_0 I_T^\alpha \beta_m(NT) = \sum_{k=0}^{m} \binom{m}{k} N^k \alpha_{m-k} \; {}_0 I_T^\alpha (T^k) = \sum_{k=0}^{m} \binom{m}{k} N^k \alpha_{m-k} \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} \mathrm{T}^{k+\alpha}. \tag{30} $$

Then, the Equ. (30) yields

$$ \frac{1}{\Gamma(\alpha)} \int_0^{t-\frac{n-1}{N}} \tau^{\alpha-1} \beta_m(N(t-\tau) - n + 1) d\tau = \sum_{k=0}^{m} \binom{m}{k} N^k \alpha_{m-k} \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} \left(t - \frac{n-1}{N}\right)^{k+\alpha}. \tag{31} $$

In addition, for $t \in [\frac{n}{N}, 1)$, by using the Equs. (24) and (28), we have

$$I_L^{\alpha} b_{nm}(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \tau^{\alpha-1} b_{nm}(t-\tau) d\tau$$
$$= \frac{1}{\Gamma(\alpha)} \int_{t-\frac{n}{N}}^{t-\frac{n-1}{N}} \tau^{\alpha-1} \beta_m(N(t-\tau)-n+1) d\tau$$
$$= \frac{1}{\Gamma(\alpha)} \left(\int_0^{t-\frac{n-1}{N}} \tau^{\alpha-1} \beta_m(N(t-\tau)-n+1) d\tau - \int_0^{t-\frac{n}{N}} \tau^{\alpha-1} \beta_m(N(t-\tau)-n+1) d\tau \right). \quad (32)$$

The first integral in relation (32) is equal to Equ. (31) and the second integral is calculated using the Equ. (23) as follows:

$$\frac{1}{\Gamma(\alpha)} \int_0^{t-\frac{n}{N}} \tau^{\alpha-1} \beta_m(N(t-\tau)-n+1)$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^{t-\frac{n}{N}} \tau^{\alpha-1} \beta_m\big(1-(-N(t-\tau)+n)\big) d\tau$$
$$= \frac{(-1)^m}{\Gamma(\alpha)} \int_0^{t-\frac{n}{N}} \tau^{\alpha-1} \beta_m\left(-N\left(\left(t-\frac{n}{N}\right)-\tau\right)\right) d\tau, \quad (33)$$

by substituting $Z = t - \frac{n}{N}$ into Equ. (33), we have

$$\frac{(-1)^m}{\Gamma(\alpha)} \int_0^Z \tau^{\alpha-1} \beta_m\big(-N(Z-\tau)\big) d\tau = (-1)^m \, {}_0I_Z^{\alpha} \beta_m(-NZ). \quad (34)$$

Following a similar approach to the one used in Equ. (30), the Equs. (32) to (34) yield

$$\frac{1}{\Gamma(\alpha)} \int_0^{t-\frac{n}{N}} \tau^{\alpha-1} \beta_m(N(t-\tau)-n+1) d\tau$$
$$= (-1)^m \sum_{k=0}^{m} \binom{m}{k} (-N)^k \alpha_{m-k} \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} \left(t-\frac{n}{N}\right)^{k+\alpha}. \quad (35)$$

Finally, using the preceding results, we can write

$$I_L^{\alpha} b_{nm}(t) = \begin{cases} 0, & t \in \left[0, \frac{n-1}{N}\right), \\ R_{nm}(t), & t \in \left[\frac{n-1}{N}, \frac{n}{N}\right), \\ R_{nm}(t) - S_{nm}(t), & t \in \left[\frac{n}{N}, 1\right), \end{cases} \quad (36)$$

Where $R_{nm}(t)$ and $S_{nm}(t)$ are the relations (31) and (35), respectively.

B. RRLFI Operator

Now, we obtain the RRLFI operator for hybrid functions. Suppose $t \in \left[\frac{n-1}{N}, \frac{n}{N}\right), n = 1,2,\dots,N$, and ${}_tI_1^{\alpha} B(t) = I_R^{\alpha} B(t)$, from Equ. (5) and Bernoulli hybrid functions (24), we have

$$I_R^{\alpha} b_{nm}(t) = \frac{1}{\Gamma(\alpha)} \int_t^1 (\tau-t)^{\alpha-1} b_{nm}(\tau) d\tau$$
$$= \frac{1}{\Gamma(\alpha)} \int_t^{\frac{n}{N}} (\tau-t)^{\alpha-1} \beta_m(N\tau-n+1) d\tau$$
$$= {}_tI_{\frac{n}{N}}^{\alpha} \beta_m(Nt-n+1) = {}_tI_{\frac{n}{N}}^{\alpha} \beta_m\left(1-N\left(\frac{n}{N}-t\right)\right).$$

From Equs. (17), (22), and (23), we get

$${}_tI_{\frac{n}{N}}^{\alpha} \beta_m(Nt-n+1) = (-1)^m \, {}_tI_{\frac{n}{N}}^{\alpha} \beta_m\left(N\left(\frac{n}{N}-t\right)\right)$$
$$= (-1)^m \, {}_tI_{\frac{n}{N}}^{\alpha} \left(\sum_{k=0}^{m} \binom{m}{k} N^k \alpha_{m-k} \left(\frac{n}{N}-t\right)^k \right)$$
$$= (-1)^m \sum_{k=0}^{m} \binom{m}{k} N^k \alpha_{m-k} \, {}_tI_{\frac{n}{N}}^{\alpha} \left(\frac{n}{N}-t\right)^k$$
$$= (-1)^m \sum_{k=0}^{m} \binom{m}{k} N^k \alpha_{m-k} \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} \left(\frac{n}{N}-t\right)^{k+\alpha}.$$

So, we conclude

$${}_tI_{\frac{n}{N}}^{\alpha} \beta_m(Nt-n+1)$$
$$= (-1)^m \sum_{k=0}^{m} \binom{m}{k} N^k \alpha_{m-k} \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} \left(\frac{n}{N}-t\right)^{k+\alpha}. \quad (37)$$

For $t \in \left[0, \frac{n-1}{N}\right)$, from Equs. (5) and (24), the RRLFI operator is obtained as follows:

$$I_R^{\alpha} b_{nm}(t) = \frac{1}{\Gamma(\alpha)} \int_t^1 (\tau-t)^{\alpha-1} b_{nm}(\tau) d\tau$$
$$= \frac{1}{\Gamma(\alpha)} \int_t^{\frac{n-1}{N}} (\tau-t)^{\alpha-1} b_{nm}(\tau) d\tau$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{\frac{n-1}{N}}^{\frac{n}{N}} (\tau-t)^{\alpha-1} b_{nm}(\tau) d\tau$$
$$+ \frac{1}{\Gamma(\alpha)} \int_{\frac{n}{N}}^{1} (\tau-t)^{\alpha-1} b_{nm}(\tau) d\tau$$
$$= \frac{1}{\Gamma(\alpha)} \int_{\frac{n-1}{N}}^{\frac{n}{N}} (\tau-t)^{\alpha-1} \beta_m(N\tau-n+1) d\tau$$
$$= \frac{1}{\Gamma(\alpha)} \int_t^{\frac{n}{N}} (\tau-t)^{\alpha-1} \beta_m(N\tau-n+1) d\tau$$
$$- \frac{1}{\Gamma(\alpha)} \int_t^{\frac{n-1}{N}} (\tau-t)^{\alpha-1} \beta_m(N\tau-n+1) d\tau$$
$$= {}_tI_{\frac{n}{N}}^{\alpha} \beta_m(Nt-n+1) - {}_tI_{\frac{n-1}{N}}^{\alpha} \beta_m(Nt-n+1).$$

${}_tI_{\frac{n}{N}}^{\alpha} \beta_m(Nt-n+1)$ is equal to Equ. (37), and ${}_tI_{\frac{n-1}{N}}^{\alpha} \beta_m(Nt-n+1)$ is calculated as follows:

$${}_tI_{\frac{n-1}{N}}^{\alpha} \beta_m(Nt-n+1) = {}_tI_{\frac{n-1}{N}}^{\alpha} \beta_m\left(-N\left(\frac{n-1}{N}-\mathrm{t}\right)\right)$$
$$= \sum_{k=0}^{m} \binom{m}{k} (-N)^k \alpha_{m-k} \, {}_tI_{\frac{n-1}{N}}^{\alpha} \left(\frac{n-1}{N}-t\right)^k,$$

By using Equ. (17), we get

$${}_tI_{\frac{n-1}{N}}^{\alpha} \beta_m(Nt-n+1)$$
$$= \sum_{k=0}^{m} \binom{m}{k} (-N)^k \alpha_{m-k} \frac{\Gamma(k+1)}{\Gamma(k+\alpha+1)} \left(\frac{n-1}{N}-t\right)^{k+\alpha}. \quad (38)$$

By applying the Equs. (37) and (38), we deduce that

$I_R^{\alpha} b_{nm}(t) =$

$$\begin{cases} {}_tI_{\frac{n}{N}}^{\alpha}\beta_m(Nt-n+1) - {}_tI_{\frac{n-1}{N}}^{\alpha}\beta_m(Nt-n+1), & t\in\left[0,\frac{n-1}{N}\right), \\ {}_tI_{\frac{n}{N}}^{\alpha}\beta_m(Nt-n+1) & t\in\left[\frac{n-1}{N},\frac{n}{N}\right), \\ 0, & t\in\left[\frac{n}{N},1\right). \end{cases} \quad (39)$$

IV. DESCRIPTION OF THE PROPOSED METHOD

In this section, we find $x(t)$ and $u(t)$ such that the constraint (2) and the initial condition (3) are satisfied and the objective function (1) of the problem becomes minimum. To this end, we transform the necessary optimality conditions of the FOCP into an equivalent fractional TPBVP. Then, using fractional operators and the collocation method, TPBVP is converted to a system of algebraic equations. By solving this system, the state variable$x(t)$, the costate variable$\lambda(t)$, and finally the control variable $u(t)$, are determined.

A. *Necessary Optimality Conditions*

In this part, we obtain the necessary optimality conditions for FOCP in the form of fractional TPBVP according to the proposed method based on the LCFD and RCFD operators.

Theorem 1 If $\big(x(t),\ u(t)\big)$ be the optimal solution of FOCP expressed in Equs. (1) to (3), there will be a costate function $\lambda(t)$, such that

$$\begin{cases} K_2\,{}_t^{C}D_{t_f}^{\alpha}\lambda(t) - K_1\dot{\lambda}(t) = P\big(t,x(t),\lambda(t)\big), \\ K_1\dot{x}(t) + K_2\,{}_{t_0}^{C}D_t^{\alpha}x(t) = Q\big(t,x(t),\lambda(t)\big), \\ x(t_0)=x_0,\ \ \lambda\big(t_f\big)=0, \end{cases} \quad (40)$$

Where P and Q are the known functions in terms of $x(t)$ and $\lambda(t)$.

Proof This theorem is stated in reference [32] based on RRLFD. For compatibility with the proposed method, we replace RRLFD with RCFD. For this purpose, from Equ. (14), RRLFD of the costate function can be written in the following form:

$${}_t^{RL}D_{t_f}^{\alpha}\lambda(t) = {}_t^{C}D_{t_f}^{\alpha}\lambda(t) + \frac{\lambda(t_f)}{\Gamma(1-\alpha)}(t_f-t)^{-\alpha},$$

since$\lambda\big(t_f\big)=0$, we apply ${}_t^{C}D_{t_f}^{\alpha}\lambda(t)$, in the first Equ. (40) instead of the RRLFD. Also, using the necessary optimality conditions, the control variable can be represented in terms of costate and state variables. □

B. *Solving the Fractional TPBVP*

In this part, the Bernoulli hybrid functions, LRLFI and RRLFI operators, and Legendre collocation method are used to approximate the solution of the mentioned fractional TPBVP. For solving the fractional differential equations (40) without loss of generality, let $t_0=0$ and $t_f=1$. Then, we expand the first-order derivative of the state function $\dot{x}(t)$ and costate function $\dot{\lambda}(t)$ by the hybrid function as follows:

$$\dot{x}(t) = A^T B(t), \quad (41)$$

$$\dot{\lambda}(t) = C^T \bar{B}(t), \quad (42)$$

where

$$B^T(t) = [b_{10}(t), b_{20}(t), \dots, b_{N0}(t), b_{11}(t), \dots, b_{N1}(t), \dots, b_{1M}(t), \dots, b_{NM}(t)], \quad (43)$$

$$A^T = [a_{10}, a_{20}, \dots, a_{N0}, a_{11}, \dots, a_{N1}, \dots, a_{1M}, \dots, a_{NM}], \quad (44)$$

$$\bar{B}^T(t) = [b_{10}(t), b_{20}(t), \dots, b_{\acute{N}0}(t), b_{11}(t), \dots, b_{\acute{N}1}(t), \dots, b_{1\acute{M}}(t), \dots, b_{\acute{N}\acute{M}}(t)], \quad (45)$$

$$C^T = [c_{10}, c_{20}, \dots, c_{\acute{N}0}, c_{11}, \dots, c_{\acute{N}1}, \dots, c_{1\acute{M}}, \dots, c_{\acute{N}\acute{M}}]. \quad (46)$$

If $\alpha=0, \alpha=1$, Equs. (4) and (41) yield

$$x(t) = A^T I_L^1 B(t) + x_0, \quad (47)$$

By applying properties (10), (12), and (15) in Equ. (47), we obtain

$${}_0^{C}D_t^{\alpha}x(t) = A^T I_L^{1-\alpha}B(t). \quad (48)$$

Let $\alpha=1, b=1$, from the Equs. (5), (42), and boundary conditions (40), we get

$$\lambda(t) = -C^T I_R^1 \bar{B}(t), \quad (49)$$

By using Equs. (10) and (15), the Equ. (49) yields

$${}_t^{C}D_1^{\alpha}\lambda(t) = -C^T I_R^{1-\alpha}\bar{B}(t). \quad (50)$$

Substituting relations (41), (42), and (47) to (50) into Equs. (40) implies that

$$\begin{aligned} &-K_1 C^T \bar{B}(t) - K_2 C^T I_R^{1-\alpha}\bar{B}(t) \\ &\quad = P\big(t, A^T I_L^1 B(t) + x_0, -C^T I_R^1 \bar{B}(t)\big), \\ &K_1 A^T B(t) + K_2 A^T I_L^{1-\alpha} B(t) \\ &\quad = Q\big(t, A^T I_L^1 B(t) + x_0, -C^T I_R^1 \bar{B}(t)\big). \end{aligned} \quad (51)$$

After simplifying the Equs. (51), if possible, now we collocate them in the Legendre-Gauss collocation points $\{t_i\}_{i=0}^{k}, k = N(M+1) + \acute{N}\big(\acute{M}+1\big) - 1$, on the interval $[0,1]$ which are the roots of Legendre polynomial p_{k+1}. The system of Equs. (51) is reduced to that of algebraic equations as follows:

$$\begin{aligned} &-K_1 C^T \bar{B}(t_i) - K_2 C^T I_R^{1-\alpha}\bar{B}(t_i) \\ &\quad = P\big(t_i, A^T I_L^1 B(t_i) + x_0, -C^T I_R^1 \bar{B}(t_i)\big), \\ &K_1 A^T B\big(t_j\big) + K_2 A^T I_L^{1-\alpha} B\big(t_j\big) \\ &\quad = Q\left(t_j, A^T I_L^1 B\big(t_j\big) + x_0, -C^T I_R^1 \bar{B}\big(t_j\big)\right), \end{aligned} \quad (52)$$

Where $i = 0,1,\dots,\acute{N}\big(\acute{M}+1\big)-1$ and $j = \acute{N}\big(\acute{M}+1\big), \dots, k$. By solving Equs. (52) via a standard numerical method, the unknown coefficient A^T and C^T will be determined.

Remark 1 In the dynamic constraint of the system, if $K_1=0$, the approximate functions of the problem-solving process are considered as follows:

$${}_0^{C}D_t^{\alpha}x(t) = A^T B(t), \quad (53)$$

$${}_t^{C}D_1^{\alpha}\lambda(t) = C^T \bar{B}(t). \quad (54)$$

By using Equs. (11), (16), and LRLFI and RRLFI operators, from Equs. (53) and (54), $x(t)$ and $\lambda(t)$ can be written as:

$$x(t) = A^T I_L^{\alpha} B(t) + x(0) = A^T I_L^{\alpha} B(t) + x_0, \quad (55)$$

$$\lambda(t) = C^T I_R^{\alpha}\bar{B}(t) + \lambda(1) = C^T I_R^{\alpha}\bar{B}(t). \quad (56)$$

V. ERROR BOUNDS

In this section, the error bounds for the hybrid functions approximation, LRLFI and RRLFI operators, and the proposed method are presented.

A. Error Bound for Hybrid Functions Approximation

In the following theorem, the upper bound for the error and the convergence of Bernoulli hybrid functions approximation are investigated.

Theorem 2 Suppose that $f(t) \in C^{(M+1)}[0,1]$ and $f_{NM}(t) = \sum_{n=1}^{N}\sum_{m=0}^{M} c_{nm} b_{nm}(t)$, be the best approximation $f(t)$ out of Y given in (25) and (26), the following inequality is satisfied:

$$\|f(t) - f_{NM}(t)\|_2 \le \frac{L}{(M+1)!N^{(M+1)}\sqrt{2M+3}}, \quad (57)$$

where

$$L = max\left|f^{(M+1)}(t)\right|, t \in [0,1]. \quad (58)$$

Proof We divide interval [0,1] into subintervals $\left[\frac{n-1}{N}, \frac{n}{N}\right], n = 1,2,\dots,N$, with the limitation that f_n approximates f over the subinterval $\left[\frac{n-1}{N}, \frac{n}{N}\right], n = 1,2,\dots,N$ and $f(t) \simeq \sum_{n=1}^{N} f_n(t)$. Let the Taylor polynomials

$$f_n(t) = \sum_{k=0}^{M} f^{(k)}\left(\frac{n-1}{N}\right)\frac{(t-\frac{n-1}{N})^k}{k!},$$

We know that

$$|f(t) - f_n(t)| \le \left|f^{(M+1)}(\xi)\right|\frac{\left(t-\frac{n-1}{N}\right)^{M+1}}{(M+1)!}, \xi \in \left[\frac{n-1}{N}, \frac{n}{N}\right]. \quad (59)$$

Assume that $f_{nM}(t) = \sum_{m=0}^{M} c_{nm} b_{nm}(t)$, from Equs. (24), (58), and (59), we have

$$\begin{aligned}\|f(t) - f_{NM}\|_2^2 &= \int_0^1 |f(t) - f_{NM}(t)|^2 dt \\ &= \sum_{n=1}^{N}\int_{\frac{n-1}{N}}^{\frac{n}{N}} |f(t) - f_{nM}(t)|^2 dt \\ &\le \sum_{n=1}^{N}\int_{\frac{n-1}{N}}^{\frac{n}{N}} |f(t) - f_n(t)|^2 dt \\ &\le \sum_{n=1}^{N}\int_{\frac{n-1}{N}}^{\frac{n}{N}} \left|f^{(M+1)}(\xi)\right|^2 \left(\frac{\left(t-\frac{n-1}{N}\right)^{M+1}}{(M+1)!}\right)^2 dt \\ &\le \frac{L^2}{(2M+3)\left((M+1)!\right)^2 N^{2(M+1)}},\end{aligned}$$

And by taking square roots, the proof is finished. This theorem shows that the Bernoulli hybrid functions approximation error tends to zero if M and N are sufficiently increased. This result confirms that f_{NM} converges to f. □

B. Error Bound for the LRLFI Operator

Theorem 3 Suppose $f(t) \in C^{(M+1)}[0,1]$ and $0 < \alpha \le 1$, the error bound for the LRLFI operator is achieved as follows:

$$\|I_L^{\alpha} f(t) - I_L^{\alpha} f_{NM}(t)\|_2 \le \frac{L}{\Gamma(\alpha+1)(M+1)!N^{(M+1)}\sqrt{2M+3}}. \quad (60)$$

Proof This theorem is proved by using inequalities (19) and (57). □

C. Error Bound for the RRLFI Operator

Theorem 4 Assume $f(t) \in C^{(M+1)}[0,1]$. For $0 < \alpha \le 1$, the error bound for the RRLFI operator is defined by:

$$\|I_R^{\alpha} f(t) - I_R^{\alpha} f_{NM}(t)\|_2 \le \frac{L}{\Gamma(\alpha+1)(M+1)!N^{(M+1)}\sqrt{2M+3}}. \quad (61)$$

Proof By using relations (20) and (57), inequality (61) is resulted. □

D. Error Bound for the Proposed Method

In this section, we estimate the error of the proposed method with respect to the hybrid functions order $N, M, \acute{N}$, and $\acute{M}$. This theorem shows while the dimensions of the basis functions are increased, the error bounds tend to zero, consequently the state and control approximate variables converge to the exact values.

Theorem 5 Suppose $x(t)$ and $\lambda(t)$ are the exact solutions of the TPBVP (40), $x(t)$ and $\lambda(t) \in C^{(M+2)}[0,1]$. $x_{NM}(t) = A^T I_L^1 B(t) + x_0$ and $\lambda_{\acute{N}\acute{M}}(t) = -C^T I_R^1 \bar{B}(t)$ are the approximate solutions of $x(t)$ and $\lambda(t)$ where achieved from Equs. (47) and (49). Also $P\big(t, x(t), \lambda(t)\big)$ and $Q\big(t, x(t), \lambda(t)\big)$ are Lipschitz functions, with the Lipschitz constants P_i, Q_i, for $i = 1, 2$, respectively. The error bounds of (40) showing with E_1 and E_2, for the proposed method are obtained as follows:

$$\begin{aligned}\|E_1\|_2 &\le \frac{P_1 L_1}{(M+1)!N^{(M+1)}\sqrt{2M+3}} + \frac{\left(K_1 + \frac{K_2}{\Gamma(2-\alpha)} + P_2\right)L_2}{(\acute{M}+1)!\acute{N}^{(\acute{M}+1)}\sqrt{2\acute{M}+3}}, \\ \|E_2\|_2 &\le \frac{\left(K_1 + \frac{K_2}{\Gamma(2-\alpha)} + Q_1\right)L_1}{(M+1)!N^{(M+1)}\sqrt{2M+3}} + \frac{Q_2 L_2}{(\acute{M}+1)!\acute{N}^{(\acute{M}+1)}\sqrt{2\acute{M}+3}},\end{aligned} \quad (62)$$

Where

$L_1 = max\left|x^{(M+2)}(t)\right|, t \in [0,1]$ and $L_2 = max\left|\lambda^{(M+2)}(t)\right|$, $t \in [0,1]$.

Proof We define

$$\begin{aligned}\|E_1\|_2 &= \Big\|K_2\, {}_t^C D_1^{\alpha}\lambda(t) - K_1\dot{\lambda}(t) - P\big(t, x(t), \lambda(t)\big) \\ &-\Big(K_2\, {}_t^C D_1^{\alpha}\lambda_{\acute{N}\acute{M}}(t) - K_1\dot{\lambda}_{\acute{N}\acute{M}}(t) - P\big(t, x_{NM}(t), \lambda_{\acute{N}\acute{M}}(t)\big)\Big)\Big\|_2,\end{aligned}$$

according to the method outlined in Section IV, we get

$$\begin{aligned}\|E_1\|_2 &= \Big\|K_2\, {}_t^C D_1^{\alpha}\lambda(t) - K_1\dot{\lambda}(t) - P\big(t, x(t), \lambda(t)\big) \\ &+K_2 C^T I_R^{1-\alpha}\bar{B}(t) + K_1 C^T \bar{B}(t) \\ &+P\big(t, A^T I_L^1 B(t) + x_0, -C^T I_R^1 \bar{B}(t)\big)\Big\|_2,\end{aligned}$$

from the Lipschitz condition and Equs. (13), we have

$$\begin{aligned}\|E_1\|_2 &\le K_1\left\|\dot{\lambda}(t) - C^T\bar{B}(t)\right\|_2 \\ &+K_2\|{}_t^C D_1^{\alpha}\lambda(t) + C^T I_R^{1-\alpha}\bar{B}(t)\|_2 \\ &+\left\|P\big(t, x(t), \lambda(t)\big) - P\big(t, A^T I_L^1 B(t) + x_0, -C^T I_R^1 \bar{B}(t)\big)\right\|_2 \\ &\le K_1\left\|\dot{\lambda}(t) - C^T\bar{B}(t)\right\|_2 + K_2\left\|-I_R^{1-\alpha}\dot{\lambda}(t) + C^T I_R^{1-\alpha}\bar{B}(t)\right\|_2 \\ &+P_1\|x(t) - (A^T I_L^1 B(t) + x_0)\|_2 + P_2\|\lambda(t) + C^T I_R^1\bar{B}(t)\|_2.\end{aligned} \quad (63)$$

From definitions (4) and (5), we have $x(t) = I_L^1\dot{x}(t) + x_0$ and $\lambda(t) = -I_R^1\dot{\lambda}(t)$. By substituting these equations into Equ. (63) and using error bounds (57), (60) and (61), we have

$$\begin{aligned}\|E_1\|_2 &\le K_1\left\|\dot{\lambda}(t) - C^T\bar{B}(t)\right\|_2 \\ &+K_2\left\|-I_R^{1-\alpha}\dot{\lambda}(t) + C^T I_R^{1-\alpha}\bar{B}(t)\right\|_2 \\ &+P_1\|I_L^1\dot{x}(t) - A^T I_L^1 B(t)\|_2 + P_2\left\|-I_R^1\dot{\lambda}(t) + C^T I_R^1\bar{B}(t)\right\|_2 \\ &\le K_1\left\|\dot{\lambda}(t) - C^T\bar{B}(t)\right\|_2 + \frac{K_2}{\Gamma(2-\alpha)}\left\|\dot{\lambda}(t) - C^T\bar{B}(t)\right\|_2\end{aligned}$$

$$+P_1\|\dot{x}(t)-A^TB(t)\|_2+P_2\left\|\dot{\lambda}(t)-C^T\bar{B}(t)\right\|_2$$
$$=P_1\|\dot{x}(t)-A^TB(t)\|_2$$
$$+\left(K_1+\frac{K_2}{\Gamma(2-\alpha)}+P_2\right)\left\|\dot{\lambda}(t)-C^T\bar{B}(t)\right\|_2$$
$$\leq\frac{P_1L_1}{(M+1)!N^{(M+1)}\sqrt{2M+3}}+\frac{\left(K_1+\frac{K_2}{\Gamma(2-\alpha)}+P_2\right)L_2}{(\acute{M}+1)!\acute{N}^{(\acute{M}+1)}\sqrt{2\acute{M}+3}}.$$

Now we define $\|E_2\|_2$ as follows:

$$\|E_2\|_2=\left\|K_1\dot{x}(t)+K_2{}_0^CD_t^{\alpha}x(t)-Q\left(t,x(t),\lambda(t)\right)\right.$$
$$\left.-\left(K_1\dot{x}_{NM}(t)+K_2{}_0^CD_t^{\alpha}x_{NM}(t)-Q\left(t,x_{NM}(t),\lambda_{\acute{N}\acute{M}}(t)\right)\right)\right\|_2,$$

similarly, we can obtain the following relation

$$\|E_2\|_2\leq\frac{\left(K_1+\frac{K_2}{\Gamma(2-\alpha)}+Q_1\right)L_1}{(M+1)!N^{(M+1)}\sqrt{2M+3}}+\frac{Q_2L_2}{(\acute{M}+1)!\acute{N}^{(\acute{M}+1)}\sqrt{2\acute{M}+3}}.\quad\square$$

VI. ILLUSTRATIVE EXAMPLES

In this section, some examples are presented to illustrate the efficiency and accuracy of the proposed method. The obtained results have been compared with those reported by using other methods.

Example 1 Consider the following two-dimensional FOCP [34]

$$\min J=\int_0^1\left((x_1(t)-1-t^{1.5})^2+(x_2(t)-t^{2.5})^2+\left(u(t)-\frac{3\sqrt{\pi}}{4}t+t^{2.5}\right)^2\right)dt,$$

subject to:

$${}_0^CD_t^{0.5}x_1(t)=x_2(t)+u(t),$$
$${}_0^CD_t^{0.5}x_2(t)=x_1(t)+\frac{15\sqrt{\pi}}{16}t^2-t^{1.5}-1,$$
$$x_1(0)=1,x_2(0)=0.$$

For this problem, $x_1(t)=1+t^{1.5}$, $x_2(t)=t^{2.5}$, and $u(t)=\frac{3\sqrt{\pi}}{4}t-t^{2.5}$ minimize the cost function and the minimum value is $J=0$. The necessary optimality conditions are as follows:

$$\begin{aligned}
&{}_0^CD_t^{0.5}x_1(t)=x_2(t)+\tfrac{3\sqrt{\pi}}{4}t-t^{2.5}-\tfrac{1}{2}\lambda_1(t),\\
&{}_0^CD_t^{0.5}x_2(t)=x_1(t)+\tfrac{15\sqrt{\pi}}{16}t^2-t^{1.5}-1,\\
&{}_t^CD_1^{0.5}\lambda_1(t)=\lambda_2(t)-2t^{1.5}+2x_1(t)-2,\\
&{}_t^CD_1^{0.5}\lambda_2(t)=\lambda_1(t)-2t^{2.5}+2x_2(t),\\
&u(t)+\tfrac{1}{2}\lambda_1(t)-\tfrac{3\sqrt{\pi}}{4}t+t^{2.5}=0,\\
&x_1(0)=1,x_2(0)=0,\lambda_1(1)=0,\lambda_2(1)=0.
\end{aligned}\quad(64)$$

By applying the proposed approach and Equs. (53) and (54), for $N=1,M=2$ and $\acute{N}=1,\acute{M}=1$, we obtain

$$\begin{aligned}
&{}_0^CD_t^{0.5}x_1(t)=A^TB(t)=a_{10}b_{10}(t)+a_{11}b_{11}(t)+a_{12}b_{12}(t),\\
&{}_0^CD_t^{0.5}x_2(t)=C^TB(t)=c_{10}b_{10}(t)+c_{11}b_{11}(t)+c_{12}b_{12}(t),\\
&{}_t^CD_1^{0.5}\lambda_1(t)=d_{10}b_{10}(t)+d_{11}b_{11}(t),\\
&{}_t^CD_1^{0.5}\lambda_2(t)=e_{10}b_{10}(t)+e_{11}b_{11}(t).
\end{aligned}\quad(65)$$

Using the Equs. (55) and (56) yield

$$\begin{aligned}
x_1(t)&=A^TI_L^{0.5}B(t)+x_1(0)\\
&=a_{10}I_L^{0.5}b_{10}(t)+a_{11}I_L^{0.5}b_{11}(t)+a_{12}I_L^{0.5}b_{12}(t)+1,\\
x_2(t)&=C^TI_L^{0.5}B(t)+x_2(0)\\
&=c_{10}I_L^{0.5}b_{10}(t)+c_{11}I_L^{0.5}b_{11}(t)+c_{12}I_L^{0.5}b_{12}(t),\\
\lambda_1(t)&=d_{10}I_R^{0.5}b_{10}(t)+d_{11}I_R^{0.5}b_{11}(t),\ \text{and}\\
\lambda_2(t)&=e_{10}I_R^{0.5}b_{10}(t)+e_{11}I_R^{0.5}b_{11}(t).
\end{aligned}\quad(66)$$

By substituting Equs. (65) to (66) into Equs. (64), and using the collocation method, we obtain a system of algebraic equations. By solving this system, the unknown coefficients are determined as follows:

$$\begin{aligned}
c_{10}&=0.55389182840797393036362224115202,\\
c_{11}&=1.66167548522392145371809335507 86,\\
c_{12}&=1.66167548522392137565553693612 23,\\
d_{10}&=0.00000000000000001646518398466862,\\
d_{11}&=-0.00000000000000009879110390801123,\\
a_{10}&=0.66467019408956860353341674532566,\\
a_{11}&=1.32934038817913725095654228919 59,\\
a_{12}&=0.00000000000000004463398968873306.\\
e_{10}&=0.00000000000000019409327747612383,\\
e_{11}&=0.00000000000000036094262159047331.
\end{aligned}$$

TABLE I
COMPARISON OF THE OPTIMAL VALUE J WITH METHODS IN [28], [33], AND [34] FOR EXAMPLE 1

Method	Optimal cost function J	Approximation order
Bernoulli operational matrix [28]	9.4157×10^{-8}	$m=8$
Epsilon-Ritz [33]	8.0027×10^{-6}	$k=8$
MADM [34]	8.4738×10^{-17}	$N=30$
Present method	1.9636×10^{-32}	$N=1,M=2$

The optimal cost function J obtained of the proposed method and Bernoulli fractional derivative operational matrix compatible with the spectral Ritz method in the direct approach [28], Epsilon-Ritz method in [33], and modified Adomian decomposition method (MADM) in [34], are displayed in Table I. This comparison shows that the hybrid functions proposed method provides a more accurate solution than the mentioned methods by taking a small number of basis functions. Compared to the method [28], the suggested method using the exact fractional operators versus the approximate operational matrix and applying the fractional calculus to impose initial conditions against the Ritz method, leading to a better solution**.** The graphs of the exact and approximate state and control variables are shown in Figs. 1 and 2, demonstrating the approximate solution matches the exact solution.

Example 2 We consider the following FOCP with variable fractional order [32]

$$\min J=\int_0^1\left(tu(t)-(\alpha+2)x(t)\right)^2dt,$$

subject to the dynamical system

$$\dot{x}(t)+{}_0^CD_t^{\alpha}x(t)=u(t)+t^2,\ x(0)=0,\ x(1)=\frac{2}{\Gamma(\alpha+3)}.$$

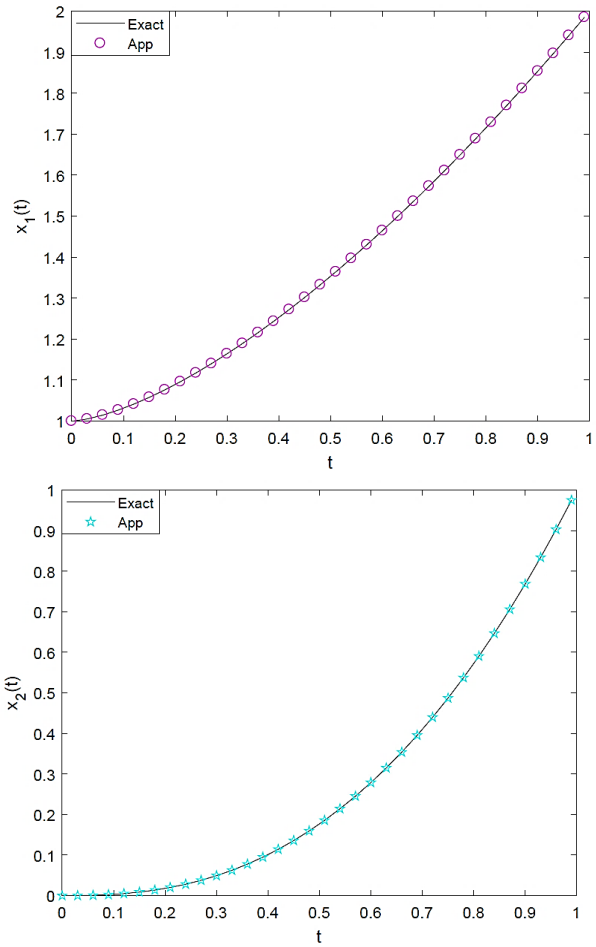


Fig. 1. The exact and approximate state variable $x_1(t)$ and $x_2(t)$ for Example 1.

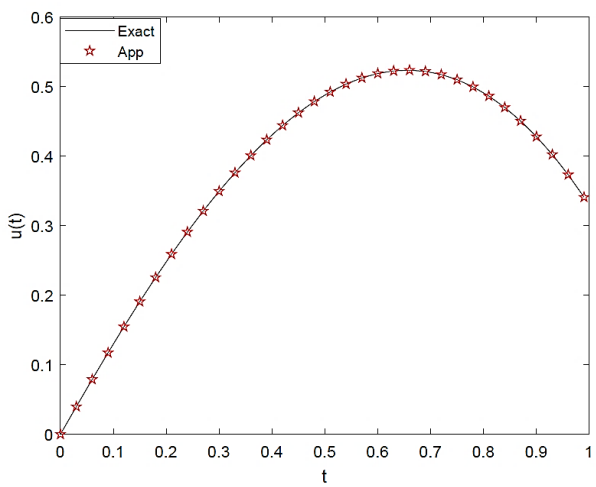


Fig. 2. The exact and approximate control variable $u(t)$ for Example 1.

For this problem, the exact solution is $\big(x(t),u(t)\big)=\left(\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)},\frac{2t^{\alpha+1}}{\Gamma(\alpha+2)}\right)$. The necessary optimality conditions are as follows:

$$2t\big(tu(t)-(\alpha+2)x(t)\big)+\lambda(t)=0, \tag{67}$$

$$\dot{\lambda}(t)-{}_t^C D_1^{\alpha}\lambda(t)=-\lambda(t)\frac{(\alpha+2)}{t},$$
$$\dot{x}(t)+{}_0^C D_t^{\alpha}x(t)=u(t)+t^2,\quad x(0)=0,\lambda(1)=0.$$

From Equ. (67), we have

$$u(t)=\frac{(\alpha+2)}{t}x(t)-\frac{\lambda(t)}{2t^2}. \tag{68}$$

Here, we approximate the unknown functions by the hybrid functions as follows:

$$\dot{\lambda}(t)=d_{10}b_{10}(t)+d_{11}b_{11}(t),$$
$$\lambda(t)=-d_{10}I_R^1 b_{10}(t)-d_{11}I_R^1 b_{11}(t), \tag{69}$$
$${}_t^C D_1^{\alpha}\lambda(t)=-d_{10}I_R^{1-\alpha}b_{10}(t)-d_{11}I_R^{1-\alpha}b_{11}(t).$$
$$\dot{x}(t)=A^T B(t)\ ,\ x(t)=A^T I_L^1 B(t),$$
$${}_0^C D_t^{\alpha}x(t)=A^T I_L^{1-\alpha}B(t).$$

We replaced Equs. (68) and (69) in necessary optimality conditions (67) and solved the resulting equations. This problem was solved in [32] by two algorithms (Alg.1 and Alg.2). The Alg.1 is based on calculating the necessary optimality conditions and solves the resulted equations using the spectral method. In Alg.2 the state function was firstly discretized using the numerical integration, followed by the Rayleigh-Ritz method to evaluate state and control functions.

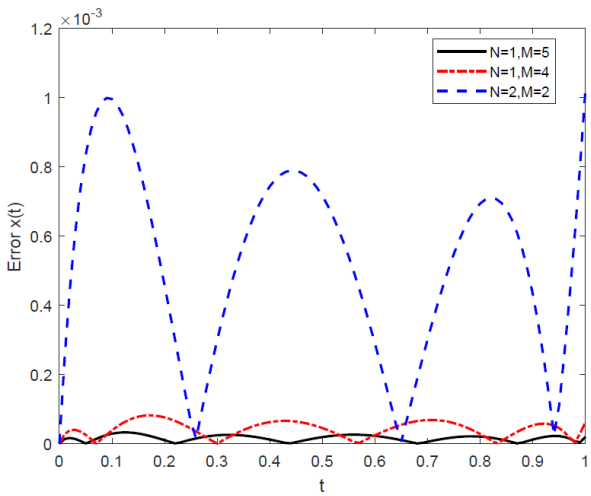


Fig. 3. Comparison of the errors $x(t)$ at $\alpha=0.5$ for different values of N and M for Example 2.

A direct approach based on Chebyshev polynomials and the Legendre-Gauss quadrature formula is employed to solve this FOCP [35]. The comparison of maximum absolute errors in the state and control variables of the present method with those of proposed numerical schemes in [32] and [35] is shown in Tables II and III. Figs. 3 and 4 display the absolute errors of $x(t)$ and $u(t)$ by selecting the different values of N and M at $\alpha=0.5$. These graphs show that the error of the solutions is decreased by increasing the number of basis functions. The exact and approximate solutions of state and control variables for different values of α are depicted in Fig. 5.

Example 3 As a practical and nonlinear example, consider the optimal maneuvers of a rigid asymmetric spacecraft. The Euler equations for the angular velocities $x_1(\tau)$, $x_2(\tau)$, and $x_3(\tau)$

of spacecraft are given by

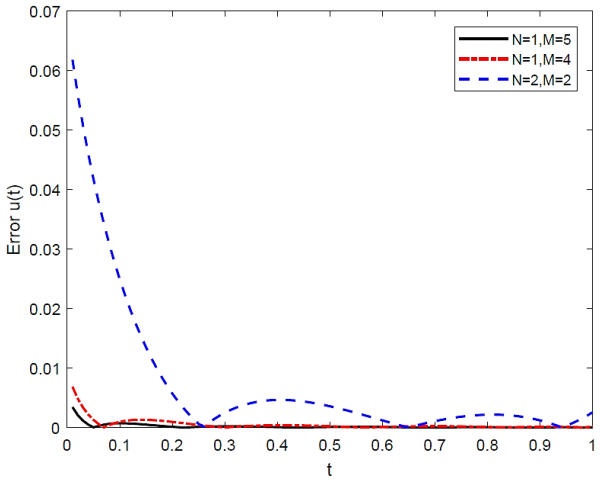


Fig. 4. Comparison of the errors $u(t)$ at $\alpha = 0.5$ for different values of N and M for Example 2.

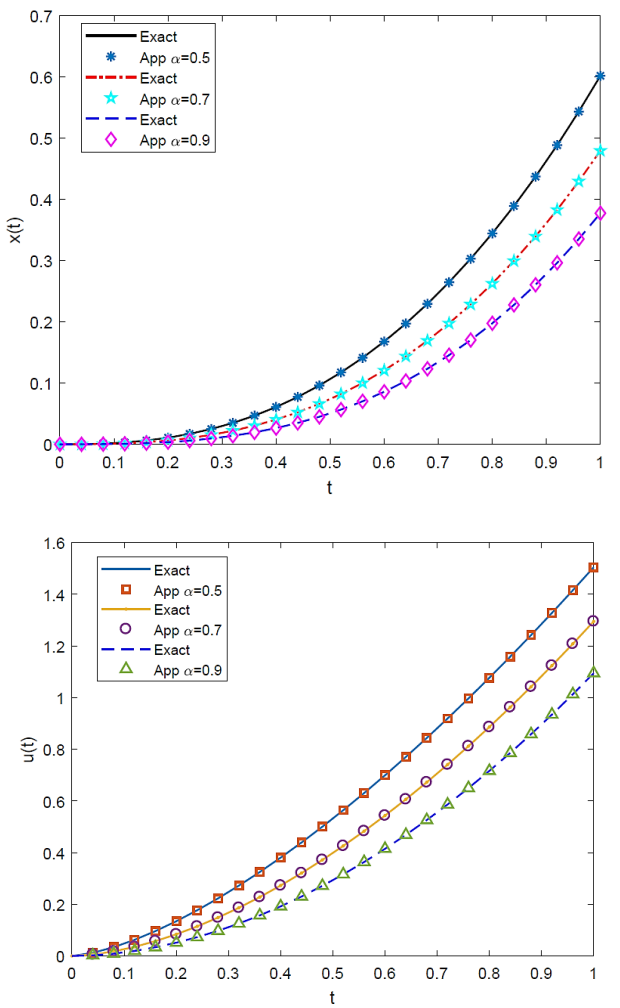


Fig. 5. The exact and approximate state and control variables for different values of α with $N = 1, M = 5$ for Example 2.

$$\dot{x_1}(\tau) = -\frac{I_3-I_2}{I_1}x_2(\tau)x_3(\tau) + \frac{u_1(\tau)}{I_1},$$
$$\dot{x_2}(\tau) = -\frac{I_1-I_3}{I_2}x_1(\tau)x_3(\tau) + \frac{u_2(\tau)}{I_2},$$
$$\dot{x_3}(\tau) = -\frac{I_2-I_1}{I_3}x_1(\tau)x_2(\tau) + \frac{u_3(\tau)}{I_3},$$

where u_1, u_2, and u_3 are control the torques. The spacecraft principle inertia are $I_1 = 86.24\ Kg\ m^2$, $I_2 = 85.07 Kg\ m^2$, and $I_3 = 113.59\ Kg\ m^2$. Also, $\frac{I_3-I_2}{I_1}$, $\frac{I_1-I_3}{I_2}$, and $\frac{I_2-I_1}{I_3}$ are the inertia difference ratios. The performance index to be minimized is expressed by

$$\min J = \frac{1}{2}\int_0^{100}\left(u_1^2(\tau) + u_2^2(\tau) + u_3^2(\tau)\right)d\tau,$$

The boundary conditions are as follows:

$x_1(0) = 0.01\ \frac{r}{s}$, $x_2(0) = 0.005\ \frac{r}{s}$, $x_3(0) = 0.001\ \frac{r}{s}$, and $x_1(100) = x_2(100) = x_3(100) = 0\ \frac{r}{s}$. According to the proposed method, the following TPBVP should be solved:

$$\dot{x_1}(\tau) = -\frac{I_3-I_2}{I_1}x_2(\tau)x_3(\tau) - \frac{\lambda_1(\tau)}{I_1^2},$$
$$\dot{x_2}(\tau) = -\frac{I_1-I_3}{I_2}x_1(\tau)x_3(\tau) - \frac{\lambda_2(\tau)}{I_2^2},$$
$$\dot{x_3}(\tau) = -\frac{I_2-I_1}{I_3}x_1(\tau)x_2(\tau) - \frac{\lambda_3(\tau)}{I_3^2},$$
$$\dot{\lambda_1}(\tau) = \frac{I_1-I_3}{I_2}x_3(\tau)\lambda_2(\tau) + \frac{I_2-I_1}{I_3}x_2(\tau)\lambda_3(\tau),$$
$$\dot{\lambda_2}(\tau) = \frac{I_3-I_2}{I_1}x_3(\tau)\lambda_1(\tau) + \frac{I_2-I_1}{I_3}x_1(\tau)\lambda_3(\tau),$$
$$\dot{\lambda_3}(\tau) = \frac{I_3-I_2}{I_1}x_2(\tau)\lambda_1(\tau) + \frac{I_1-I_3}{I_2}x_1(\tau)\lambda_2(\tau).$$

Also, the optimal control variables are obtained as follows:

$$u_1(\tau) = -\frac{\lambda_1(\tau)}{I_1}, u_2(\tau) = -\frac{\lambda_2(\tau)}{I_2}, and\ u_3(\tau) = -\frac{\lambda_3(\tau)}{I_3}.$$

We use transformation $\tau = 100t, t \in [0,1]$ to apply the proposed method. By using this method with $N = 1, M = 7$, we get:

$$x_1(t) = -2.468260116060948 \times 10^{-16}t^8 - 3.823003555219281 \times 10^{-14}t^7 + 2.722028451181812 \times 10^{-13}t^6 + 2.386478651140519 \times 10^{-10}t^5 - 1.195968612517397 \times 10^{-9}t^4 - 8.248480368393749 \times 10^{-7}t^3 + 2.479330170240148 \times 10^{-6}t^2 - 0.010001653525046t + 0.01,$$

$$x_2(t) = 3.957240843616689 \times 10^{-16}t^8 - 8.933546050793784 \times 10^{-15}t^7 + 5.561002887595278 \times 10^{-14}t^6 - 1.406589411719365 \times 10^{-10}t^5 + 7.027484806347674 \times 10^{-10}t^4 + 1.606375494884975 \times 10^{-6}t^3 - 4.821937043751208 \times 10^{-6}t^2 - 0.004996785000588t + 0.005,$$

$$x_3(t) = 1.707020832761955 \times 10^{-12}t^8 - 6.071238553944105 \times 10^{-12}t^7 + 7.687249774825881 \times 10^{-12}t^6 - 3.529343339794504 \times 10^{-11}t^5 + 1.550767918419068 \times 10^{-10}t^4 + 2.572588863294985 \times 10^{-7}t^3 - 7.7239218234617 \times 10^{-7}t^2 - 9.99484989810374 \times 10^{-4}t + 0.001.$$

By substituting $t = 0$ and $t = 1$, boundary conditions are obtained. In Table IV, a comparison is created among the numerical results of the cost function J, generated by the

proposed hybrid functions method by taking $N = 1$, $M = 7$ with the reported results in [36] by applying fractional order Chebyshev functions, [37] by using a quasilinearization technique based on the Chebyshev polynomials, and [38] by adopting Fibonacci wavelets and the Galerkin method. These results demonstrate the accuracy and efficiency of the proposed approach in comparison with mentioned methods. State and control approximate variables are shown in Figs. 6 and 7.

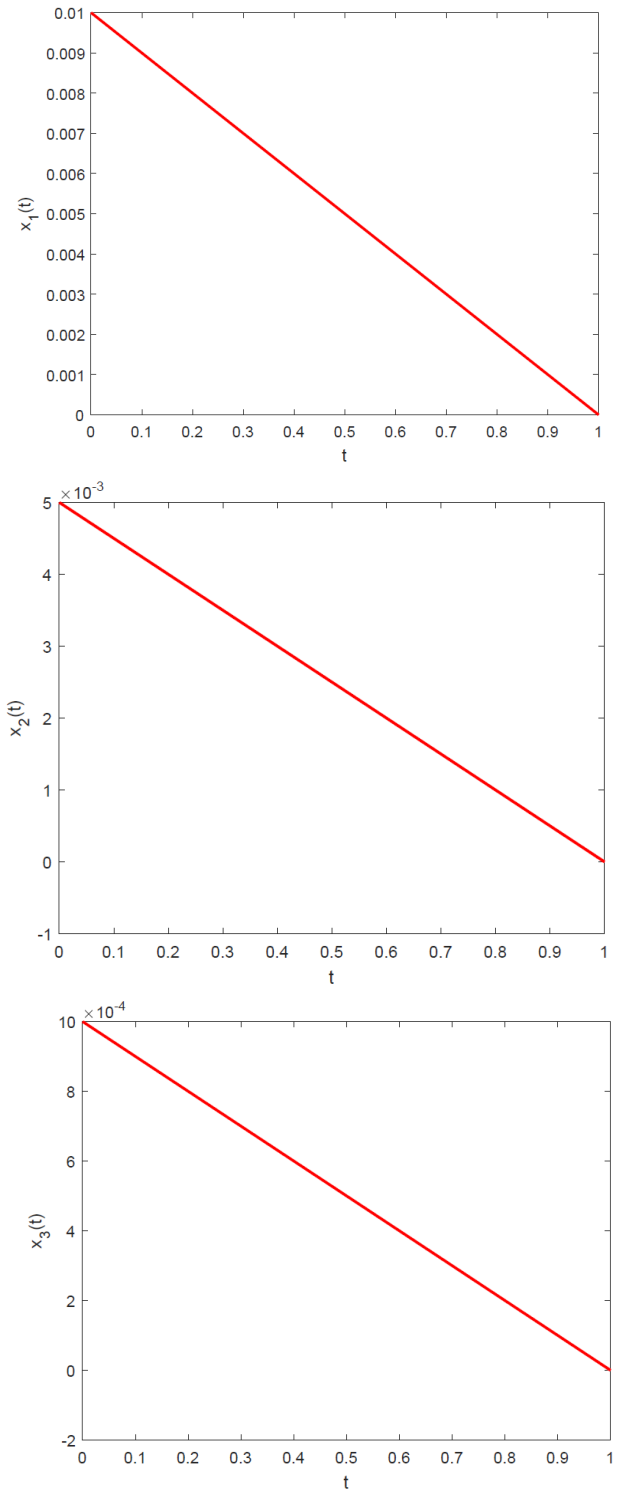


Fig. 6. The numerical values of state variables for Example 3.

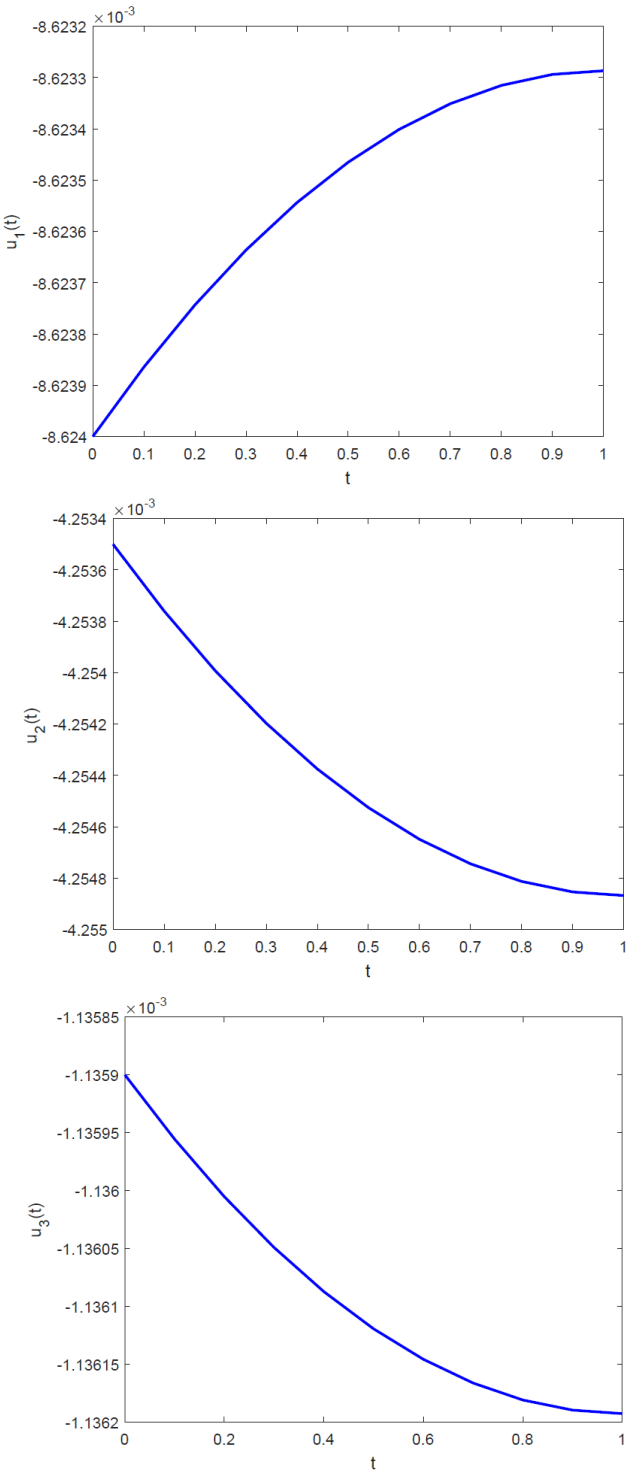


Fig. 7. The estimate values of control variables in Example 3.

TABLE II

THE MAXIMUM ERROR OF $x(t)$ AT $\alpha = 0.5$ AND COMPARISON WITH OTHER METHODS FOR EXAMPLE 2

	Alg.1[32]	Alg.2 [32]	Chebyshev polynomials [35]	Present method
$N = 2$	3.10×10^{-2}	2.73×10^{-2}	2.16×10^{-2}	1.02×10^{-3} $(N = 2, M = 2)$
$N = 5$	3.55×10^{-4}	1.60×10^{-4}	1.72×10^{-4}	3.26×10^{-5} $(N = 1, M = 5)$

TABLE III

THE MAXIMUM ERROR OF $u(t)$ AT $\alpha = 0.5$ AND COMPARISON WITH OTHER METHODS FOR EXAMPLE 2

	Alg.1[32]	Alg.2 [32]	Chebyshev polynomials [35]	Present method
$N = 2$	2.04×10^{-1}	2.56×10^{-1}	2.44×10^{-1}	6.18×10^{-2} $(N = 2, M = 2)$
$N = 5$	9.13×10^{-3}	8.22×10^{-3}	6.26×10^{-3}	3.44×10^{-3} $(N = 1, M = 5)$

TABLE IV

COMPARISON OF THE OPTIMAL VALUE J WITH METHODS IN [36], [37], AND [38] FOR EXAMPLE 3

Method	Optimal cost function J	Approximation order
Fractional Chebyshev functions [36]	0.004687795	$m = 10$
Quasi linearization [37]	0.00534063	$N = 10$
Fibonacci wavelets and Galerkin [38]	0.00526040	$k = 2, M = 4$
Present method	0.004687795353	$N = 1, M = 7$

VII. CONCLUSION

In this paper, the Bernoulli hybrid functions indirect method is presented to solve integer-fractional OCPs. Here, the LRLFI and RRLFI operators are computed for mentioned hybrid functions directly and without any approximation. By determining the necessary optimality conditions, the solution of the FOCP is transformed into solving TPBVP, including a system of FDEs. Then, the resulted system is solved, using the hybrid functions approximation and LRLFI and RRLFI operators as well as the collocation method. The error bounds and convergence of the proposed method are investigated. Finally, the method is illustrated by some test problems. The obtained numerical results are compared with the exact solutions and some of the ones available to display the accuracy and proficiency of the proposed method. As can be seen, with a few numbers of the hybrid basis functions, satisfactory results are obtained.

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