International Journal of Industrial Electronics, Control and Optimization © 2020 IECO
Vol. 3, No. 4, pp. 475-482, Sep (2020)

Novel Stability Criteria for Piecewise Affine Systems with Time-Varying Delay

Salehe Afsharian[1], Hussein Eliasi[2,†]

[1,2] Faculty of Electrical and Computer Engineering, University of Birjand, Birjand, Iran

ABSTRACT

This article aims to derive new sufficient conditions to guarantee the stability of piecewise affine systems with time-varying delay (PWA-TVD). The set of delay-dependent linear matrix inequality (LMI) describes the novel stability criteria. This approach considers the PWA-TVD system with a time-delayed state-dependent switching signal. The newly suggested Lyapunov-Krasovskii functional (L-K-F) and improved estimation of its derivative have a crucial role in decreasing the complexity and conservatism of the proposed stability results. The suggested L-K-F belongs to the current and time-delayed states, the integral of the states over the time-varying delay, and time derivation of the states. A new inequality was used to obtain an upper bound (UB) for the time derivation of the Lyapunov functional. Then based on this UB, less conservative results are achieved. The theoretical results are applied to the numerical examples. The results confirm the effectiveness of the presented method. The conservative index is the maximum admissible UB of time delay.

Article Info

Keywords:
Linear matrix inequality (LMI),Lyapunov-krasovskii functional, PWA systems, Stability analysis, Time-varying delay,

Article History:
Received 2020-01-19
Accepted 2020-04-19

I. INTRODUCTION

A piecewise affine (PWA) system is a dynamical system that contains multiple affine subsystems and a switching signal that allows switching between various subsystems. These systems have broad applications in a range of science and engineering. This type of system constructs a robust structure for to model a wide variety of hybrid and nonlinear systems in that nonlinearities are estimated by a set of linear-affine models around different operating points. [1].

A time delay is inevitable in the real world. It exists in industrial processes, control systems, energy systems, and so on. The time-delay may cause unsatisfactory efficiency, undesired oscillation, and also system instability. So, it is imperative to analyze the stability criterion of systems with time-delay [2].

The PWA systems with time-delay have a key role in the modeling of such systems as communication networks, highway transportation systems, automotive clutch systems, etc. Delay is an outstanding design and performance characteristic of these systems and cannot be ignored.

To study the stability of switching systems and PWA systems, the primary approach is to obtain a common Lyapunov function for the system under arbitrary switching [3]. For less conservative results, multiple Lyapunov functions have been proposed [4] - [7]. LyapunovKrasovskii functional and Lyapunov-Razumikhin functional techniques are used for time-delay systems to obtain stability criteria [8]. A critical goal in the delay-dependent stability analyses is to improve conditions in which one guarantees the stability for the allowable UB of time delay as large as possible. The conservativeness of the results depends on two points. The first one is the choice of proper Lyapunov functional. In order to improve the results, an augmented vector has been used to make a Lyapunov functional. Augmented state variables contain information about current and delayed states, their integral and their derivative. The stability of time-delay systems depends on the current states and their history, so the

†Corresponding Author: h_eliasi@birjand.ac.ir, Tel: +98-9122454027, Faculty of Electrical and Computer Engineering, University of Birjand, Birjand, Iran

use more states information in Lyapunov functional can be a less conservative way for stability analysis [9-12]. In all of these work, the novel augmented L-K-F are proposed, and according to this selection, the results will be less conservative. The other one is to estimate the bound of integral that appears in the time derivation of the Lyapunov functional. Several remarkable approaches have been reported to estimate a tighter bound of integral terms such as Wirtinger's inequality [13], free-weighting matrix [14], Jensen's inequality [2], double integral inequality [15], delay-dependent-matrix based (DDMB) reciprocally convex inequality [16], and etc.

The first time the PWA with time-delay systems to stability analysis has been discussed in [17]. Then the sufficient conditions that guarantee the stability of the PWA-TVD system have been investigated in [18-19]. In which, switching law is based on only the current states, and the results are more conservative. In [20-21], the state-dependent switching condition depends on the delayed states that are considered, but the delay is constant. In [22], stability conditions are derived for PWA-TVD systems with the switching law, which is dependent on the statesdependent delay and the delayed-states. Then a switching signal that is independent of the timedelayed states for the Lyapunov function is designed. In all of these works, LMIs depend only on the bound of delay and bound of its derivative, and therefore, the results are more conservative.

In the present work, the PWA-TVD systems with switching law, which is based on both the states and delayed states, are considered. We aim at improving the existing results by proposing a proper L-K-F and developing a new inequality used to derive a more accurate estimation of the lower bound of the integral, which appears in the time derivation of the L-K-F. The proposed L-K-F contains more information about the states. This Lyapunov functional depends on the current and delayed states, the integral of the states across the time-delay, and the time derivative of the states. Integral terms with time-varying delay intervals are considered in the suggested Lyapunov functional, and new stability conditions are presented, which depend on time-varying delay, and its derivative does not depend only on the bound of these. Also, we use new inequality to find the bound for the derivative of the Lyapunov functional, and we then propose stability criteria, which enhance the feasible region of stability, for the TVD-PWA system. The new stability results for the TVD-PWA system are formulated in terms of LMIs. These LMIs depend on time-varying delay and derivative of delay.

The paper is organized as below. The preliminaries and problem formulation are defined in in Section II. The main stability theorem is proposed in Section III. Section IV is dedicated to numerical examples. The paper is terminated with a conclusion paragraph in Section V.

Notation: $\mathbb{R}^{q\times r}$ is the set of all q×r real matrices. $\mathbb{S}^r$ denotes a set of symmetric r×r matrices. In addition Q > 0, for $Q\in\mathbb{R}^{n\times n}$, means that Q is positive definite. We describe $\mathrm{Sym}(X) = X + X^T$, for any square matrix $X\in\mathbb{R}^{n\times n}$. The notation I_d represents the appropriate unit matrix. The symbol * represents the symmetric structure. $K^{\perp}$ denotes the matrix basis for the null space of $K \in \mathbb{R}^{n\times m}$.

II. PROBLEM FORMULATION

The following formulation represents the PWA-TVD system:

$$S_1:\begin{Bmatrix}\dot{x}=A_i x(t)+A_i^d x_d(t)+a_i;\ \ x(t)=\sigma(t),\ t\epsilon[-h_d,0] \\ \forall\begin{bmatrix}x\\x_d\end{bmatrix}\epsilon\chi_i,\ i\in I\end{Bmatrix} \quad (1)$$

where $x(t)\in\mathbb{R}^{n\times 1}$ is the system current states, $x_d(t)=x(t-d_h(t))$ is the delayed-states, $A_i, A_i^d \in\mathbb{R}^{n\times n}$ are the current state and time-delayed state matrices respectively, $a_i\in\mathbb{R}^{n\times 1}$ is the affine terms, and $\sigma(t)$ is an initial condition. The state space is partitioned to a number of polyhedral cell that are shown with $\chi_i=\left\{(a,b)\in R^n\times R^n \,\middle|\, \bar{E}_i\begin{bmatrix}a\\1\end{bmatrix}\geq 0,\ \bar{E}_i^d\begin{bmatrix}b\\1\end{bmatrix}\geq 0\right\}$, where $\bar{E}_i=[E_i\quad e_i]$, $\bar{E}_i^d=[E_i^d\quad e_i^d]$ are the cell boundaries matrices. $I=\{1,\ldots,\Xi\}$ shows the index of the subsystems. $d_h(t)$ is a time-varying delay, which value and its rate are bounded:

$$0\leq d_h(t)\leq h_d,\ -\mu\leq\dot{d}_h(t)\leq\mu \quad (2)$$

The state space can be partitioned as below:

$$[\bar{E}_i\quad \bar{E}_i^d]\begin{bmatrix}\bar{x}(t)\\\bar{x}_d(t)\end{bmatrix}\geq 0;\ \forall\begin{bmatrix}x\\x_d\end{bmatrix}\epsilon\chi_i,\ i\in I \quad (3)$$

where $\bar{x}(t)\triangleq[x(t)\quad 1]^T$, and $\bar{x}_d(t)\triangleq[x_d(t)\quad 1]^T$.

The continuity matrices $\bar{F}_i=[F_i\quad f_i]$, $\bar{F}_i^d=[F_i^d\quad f_i^d]$, satisfying:

$$[\bar{F}_i\quad \bar{F}_i^d][\bar{x}(t)\quad \bar{x}_d(t)]^T=[\bar{F}_j\quad \bar{F}_j^d][\bar{x}(t)\quad \bar{x}_d(t)]^T \quad \forall\begin{bmatrix}x\\x_d\end{bmatrix}\epsilon\chi_i\cap\chi_j,\ i,j\ \epsilon I \quad (4)$$

The following in [23] can construct these matrices.

Define the following augmented matrices; $\bar{A}_i=\begin{bmatrix}A_i & a_i\\0 & 0\end{bmatrix}$, $\bar{A}_i^d=\begin{bmatrix}A_i^d & o\\0 & 0\end{bmatrix}$, thus the system (1) is formulated as:

$$S_2:\left\{\dot{\bar{x}}(t)=\bar{A}_i\bar{x}(t)+\bar{A}_i^d\bar{x}_d(t);\ \forall\begin{bmatrix}x\\x_d\end{bmatrix}\epsilon\chi_i,\ i\in I\right\} \quad (5)$$

The subsystems that contain the origin is expressed by I_0, and other subsystems are denoted by I_1.

III. MAIN RESULT

In this section, the set of delay-dependent LMI-based sufficient conditions is extracted to analyze the stability of the PWA-TVD. We introduce the following lemmas that will be used in the extraction of the stability criteria. Then, the main theorem is presented.

Lemma 1 [24]: If y is a differentiable function:$[\vartheta_1,\vartheta_2]\to R^n$, for any matrices $M_2\in\mathbb{R}^{3n\times 3n}$and $V_1, V_2\in\mathbb{R}^{3n\times n}$and symmetric matrices $M_1, M_3\in\mathbb{R}^{3n\times 3n}$, and $Q\in\mathbb{R}^{n\times n}$, and satisfying:

$$\begin{bmatrix} M_1 & M_2 & V_1 \\ * & M_3 & V_2 \\ * & * & Q \end{bmatrix}\geq 0 \quad (6)$$

the following inequality holds:

$$-\int_{\vartheta_1}^{\vartheta_2}\dot{y}^T(s)Q\dot{y}(s)ds\leq\varpi_1^T(\vartheta_1,\vartheta_2)\psi_1\varpi_1(\vartheta_1,\vartheta_2) \quad (7)$$

Where

$$\varpi_1(\vartheta_1,\vartheta_2)=\left[y^T(\vartheta_1) \quad y^T(\vartheta_2) \quad \frac{1}{\vartheta_1-\vartheta_2}\int_{\vartheta_1}^{\vartheta_2}y^T(s)ds\right]$$

$$\psi_1=(\vartheta_2-\vartheta_1)\left(M_1+\frac{1}{3}M_3\right)+ \mathrm{Sym}\{V_1[I_d \quad -I_d \quad 0]+V_2[-I_d \quad -I_d \quad 2I_d]\}$$

Lemma 2 [25]: (Finsler's Lemma) Let $W\in\mathbb{R}^n$, $S\in\mathbb{S}^n$, and $K\in\mathbb{R}^{m\times n}$ so that Rank (K) < n. The statements (1) and (2) are equivalent to:

(1) $W^TS\,W<0, \forall KW=0, x\neq 0$

(2) $K^{\perp T}S\,K^{\perp}<0$;

Finsler's lemma can be used to give novel linear matrix inequality characterizations to stability and control problems.

We now present novel stability criteria for the PWA-TVD system, which conditions are dependent of time-delay.

Theorem 1: For positive constants h_d and μ, the system (1), in the absence of sliding behavior, is asymptotically stable, if there exist, positive symmetric matrices R, S, Q, Z, Z_1 , Z_2, Z_3, $Z_4(\forall i\in I_0)$, $\bar{R},\bar{S},\bar{Q},\bar{Z},\bar{Z}_1,\bar{Z}_2,\bar{Z}_3,\bar{Z}_4(\forall i\in I_1)$, any matrices$Y_1$, Y_2 $(\forall i\in I_0)$, $\bar{Y}_1$, $\bar{Y}_2$ $(\forall i\in I_1)$, N_1, N_2, N_3, N_4 $(\forall i\in I_0)$, $\bar{N}_1,\bar{N}_2,\bar{N}_3,\bar{N}_4$ $(\forall i\in I_1)$, symmetric matrices $L_1\in\mathbb{R}^{m\times m}$, $H_i\in\mathbb{R}^{n\times n}$ $(i\in I)$ such that H_i have nonnegative entries, where:

$$P_i=\begin{bmatrix} F_i^TL_1F_i & F_i^TL_1F_i^d \\ F_i^{dT}L_1F_i & F_i^{dT}L_1F_i^d \end{bmatrix}, \forall i\in I_0;$$

$$\bar{P}_i=\begin{bmatrix} \bar{F}_i^TL_1\bar{F}_i & \bar{F}_i^TL_1\bar{F}_i^d \\ \bar{F}_i^{dT}L_1\bar{F}_i & \bar{F}_i^{dT}L_1\bar{F}_i^d \end{bmatrix}, \forall i\in I_1$$

,which satisfy the following LMIs for every region and for $\forall d_h(t)\in\{0,h_d\}$, $\dot{d}_h(t)\in\{-\mu,\mu\}$.

For $\forall i\in I_0$

$$\begin{bmatrix} Z_1 & Y_1 & N_1 \\ * & Z_2 & N_2 \\ * & * & R \end{bmatrix}\geq 0, \begin{bmatrix} Z_3 & Y_2 & N_3 \\ * & Z_4 & N_4 \\ * & * & R \end{bmatrix}\geq 0; \quad (8)$$
$$Z_{1,2,3,4}\in\mathbb{R}^{3n\times 3n}, N_{1,2,3,4}\in\mathbb{R}^{3n\times n}, Y_{1,2}\in R^{3n\times 3n}, \ R\in\mathbb{R}^{n\times n}$$

$$P_i-\begin{bmatrix} E_i^TH_iE_i & E_i^TH_iE_i^d \\ E_i^{dT}H_iE_i & E_i^{dT}H_iE_i^d \end{bmatrix}>0 \quad (9)$$

$$(B_i^{\perp})^T\psi_i(B_i^{\perp})<0 \quad (10)$$

where

$B_i=[A_i \quad A_i^d \quad 0 \quad -I_d \quad 0 \quad 0 \quad 0 \quad 0]$; $h_t(t)=1-\dot{d}_h(t)$,

$b_j(j=1,\dots,8)\in R^{8n\times n}$:block entry matrices, $b_0=0_{8n\times n}$;

$\psi_i=\mathrm{Sym}\{E_1P_iE_2^T\}+\mathrm{Sym}\{E_3SE_4^T\}+\Omega_1+\Omega_2+h_de_4Re_4^T+\Phi_1+\Phi_2$

$\Omega_1=E_5QE_5^T-h_tE_6QE_6^T+\mathrm{Sym}\{E_7QE_8^T\}$;

$\Omega_2=h_tE_9ZE_9^T-E_{10}ZE_{10}^T+\mathrm{Sym}\{E_{11}ZE_{12}^T\}$

$E_1=[b_1 \quad b_2]$, $E_2=[b_4 \quad h_tb_5]$, $E_3=[b_3 \quad d_h(t)b_7 \quad (h_d-d_h(t))b_8]$,

$E_4=[b_6 \quad b_1-h_tb_2 \quad h_tb_2-b_3]$,

$E_5=[b_1 \quad b_4 \quad b_0]$, $E_6=[b_2 \quad b_5 \quad b_1-b_2]$,

$E_7=[d_h(t)b_7 \quad b_1-b_2 \quad d_h(t)(b_1-b_7)]$,

$E_8=[b_0 \quad b_0 \quad b_4]$, $E_9=[b_2 \quad b_5 \quad b_0]$, $E_{10}=[b_3 \quad b_6 \quad b_2-b_3]$,

$E_{11}=[(h_d-d_h(t))b_8 \quad b_2-b_3 \quad (h_d-d_h(t))(b_2-b_8)]$,

$E_{12}=[b_0 \quad b_0 \quad h_tb_5]$, $E_{13}=[b_1 \quad b_2 \quad b_7]$, $E_{14}=[b_2 \quad b_3 \quad b_8]$

$$\Phi_1=d_h(t)E_{13}\left(Z_1+\frac{1}{3}Z_2\right)E_{13}^T+ \mathrm{Sym}\{E_{13}(N_1(b_1-b_2)^T+N_2(2b_7-b_1-b_2)^T\}$$

$$\Phi_2=(h_d-d_h(t))E_{14}\left(Z_3+\frac{1}{3}Z_4\right)E_{14}^T+ \mathrm{Sym}\{E_{14}(N_3(b_2-b_3)^T+N_4(2b_8-b_2-b_3)^T\}$$

$$S=\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12}^T & S_{22} & S_{23} \\ S_{13}^T & S_{23}^T & S_{33} \end{bmatrix}, Q=\begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}^T & Q_{22} & Q_{23} \\ Q_{13}^T & Q_{23}^T & Q_{33} \end{bmatrix},$$

$$Z=\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{12}^T & Z_{22} & Z_{23} \\ Z_{13}^T & Z_{23}^T & Z_{33} \end{bmatrix}$$

$R\in\mathbb{R}^{n\times n}$, $S\in\mathbb{R}^{3n\times 3n}$,$Q\in R^{3n\times 3n}$, $Z\in\mathbb{R}^{3n\times 3n}$

For $\forall i\in I_1$

$$\begin{bmatrix} \bar{Z}_1 & \bar{Y}_1 & \bar{N}_1 \\ * & \bar{Z}_2 & \bar{N}_2 \\ * & * & \bar{R} \end{bmatrix}\geq 0, \begin{bmatrix} \bar{Z}_3 & \bar{Y}_2 & \bar{N}_3 \\ * & \bar{Z}_4 & \bar{N}_4 \\ * & * & \bar{R} \end{bmatrix}\geq 0 \quad (11)$$
$$\bar{Z}_{1,2,3,4}\in\mathbb{R}^{3(n+1)\times 3(n+1)}, \bar{N}_{1,2,3,4}\in\mathbb{R}^{3(n+1)\times(n+1)}, \bar{Y}_{1,2}\in\mathbb{R}^{3(n+1)\times 3(n+1)}, \bar{R}\in\mathbb{R}^{(n+1)\times(n+1)}$$

$$\bar{P}_i-\begin{bmatrix} \bar{E}_i^TH_i\bar{E}_i & \bar{E}_i^TH_i\bar{E}_i^d \\ \bar{E}_i^{dT}H_i\bar{E}_i & \bar{E}_i^{dT}H_i\bar{E}_i^d \end{bmatrix}>0 \quad (12)$$

$$(\bar{B}_i^{\perp})^T\bar{\psi}_i(\bar{B}_i^{\perp})<0 \quad (13)$$

where

$\bar{B}_i=[\bar{A}_i \quad \bar{A}_i^d \quad 0 \quad -I_d \quad 0 \quad 0 \quad 0 \quad 0]$,

$\bar{b}_j(j=1,2,..,8)\in R^{8(n+1)\times(n+1)}$, $\bar{b}_0=0_{8(n+1)\times(n+1)}$

$\bar{\psi}_i=\mathrm{Sym}\left\{\bar{E}_1\bar{P}_i\bar{E}_2^T\right\}+\mathrm{Sym}\left\{\bar{E}_3\bar{S}\,\bar{E}_4^T\right\}+\bar{\Omega}_1+\bar{\Omega}_2+h_d\bar{e}_4\bar{R}\bar{e}_4^T+\bar{\Phi}_1+\bar{\Phi}_2$

$$\bar{\Omega}_1=\bar{E}_5\bar{Q}\bar{E}_5^T-h_t\bar{E}_6\bar{Q}\,\bar{E}_6^T+Sym\left\{\bar{E}_7\bar{Q}\,\bar{E}_8^T\right\}$$
$$\bar{\Omega}_2=h_t\bar{E}_9\bar{Z}\,\bar{E}_9^T-\bar{E}_{10}\bar{Z}\bar{E}_{10}^T+Sym\left\{\bar{E}_{11}\bar{Z}\,\bar{E}_{12}^T\right\}$$
$$\bar{E}_1=[\bar{b}_1 \quad \bar{b}_2],\ \bar{E}_2=[\bar{b}_4 \quad h_t\bar{b}_5],$$
$$\bar{E}_3=[\bar{b}_3 \quad d_h(t)\bar{b}_7 \quad (h_d-d_h(t))\bar{b}_8],\ \bar{E}_4=[\bar{b}_6 \quad \bar{b}_1-h_t\bar{b}_2 \quad h_t\bar{b}_2-\bar{b}_3],$$
$$\bar{E}_5=[\bar{b}_1 \quad \bar{b}_4 \quad \bar{b}_0],\ \bar{E}_6=[\bar{b}_2 \quad \bar{b}_5 \quad \bar{b}_1-\bar{b}_2]$$
$$\bar{E}_7=[d_h(t)\bar{b}_7 \quad \bar{b}_1-\bar{b}_2 \quad d_h(t)(\bar{b}_1-\bar{b}_7)]$$
$$\bar{E}_8=[\bar{b}_0 \quad \bar{b}_0 \quad \bar{b}_4],\ \bar{E}_9=[\bar{b}_2 \quad \bar{b}_5 \quad \bar{b}_0],\ \bar{E}_{10}=[\bar{b}_3 \quad \bar{b}_6 \quad \bar{b}_2-\bar{b}_3]$$
$$\bar{E}_{11}=[(h_d-d_h(t))\bar{b}_8 \quad \bar{b}_2-\bar{b}_3 \quad (h_d-d_h(t))(\bar{b}_2-\bar{b}_8)],$$
$$\bar{E}_{12}=[\bar{b}_0 \quad \bar{b}_0 \quad h_t\bar{b}_5],$$
$$\bar{E}_{13}=[\bar{b}_1 \quad \bar{b}_2 \quad \bar{b}_7],\ \bar{E}_{14}=[\bar{b}_2 \quad \bar{b}_3 \quad \bar{b}_8]$$
$$\bar{\Phi}_1=d_h(t)\bar{E}_{13}\left(\bar{Z}_1+\frac{1}{3}\bar{Z}_2\right)\bar{E}_{13}^T+ Sym\{\bar{E}_{13}(\bar{N}_1(\bar{b}_1-\bar{b}_2)^T+\bar{N}_2(2\bar{b}_7-\bar{b}_1-\bar{b}_2)^T\}$$
$$\bar{\Phi}_2=(h_d-d_h(t))\bar{E}_{14}\left(\bar{Z}_3+\frac{1}{3}\bar{Z}_4\right)\bar{E}_{14}^T+ Sym\{\bar{E}_{14}(\bar{N}_3(\bar{b}_2-\bar{b}_3)^T+\bar{N}_4(2\bar{b}_8-\bar{b}_2-\bar{b}_3)^T\}$$

$$\bar{R}=\begin{bmatrix}R & R_{12}\\ R_{12}^T & R_{22}\end{bmatrix}>0,\ \bar{S}=\begin{bmatrix}S_{11} & 0 & S_{12} & 0 & S_{13} & 0\\ * & 0 & 0 & 0 & 0 & 0\\ * & * & S_{22} & 0 & S_{23} & 0\\ * & * & * & 0 & 0 & 0\\ * & * & * & * & S_{33} & 0\\ * & * & * & * & * & 0\end{bmatrix}>0$$
$$\bar{Q}=\begin{bmatrix}Q_{11} & 0 & Q_{12} & Q_{121} & Q_{13} & Q_{131}\\ * & 0 & 0 & 0 & 0 & 0\\ * & * & Q_{22} & Q_{221} & Q_{23} & Q_{213}\\ * & * & * & Q_{222} & Q_{231}^T & Q_{232}\\ * & * & * & * & Q_{33} & Q_{331}\\ * & * & * & * & * & Q_{332}\end{bmatrix},$$
$$\bar{Z}=\begin{bmatrix}Z_{11} & 0 & Z_{12} & Z_{121} & Z_{13} & Z_{131}\\ * & 0 & 0 & 0 & 0 & 0\\ * & * & Z_{22} & Z_{221} & Z_{23} & Z_{231}\\ * & * & * & Z_{222} & Z_{231}^T & Z_{232}\\ * & * & * & * & Z_{33} & Z_{331}\\ * & * & * & * & * & Z_{332}\end{bmatrix}$$
$$\bar{S}\in\mathbb{R}^{3(n+1)\times 3(n+1)},\bar{Q}\in\mathbb{R}^{3(n+1)\times 3(n+1)},\ \bar{Z}\in\mathbb{R}^{3(n+1)\times 3(n+1)}$$

Proof: For $\forall i\in I_0$, the L-K-F is proposed as follow:

$$V_i(t,x,\dot{x})=V_{1i}(t,x)+\sum_{g=2}^{5}V_g(t,x,\dot{x})$$
$$V_{1i}(t,x)=\eta_1^T(t)P_i\eta_1(t),\ \ V_2(t,x)=\eta_2^T(t)S\eta_2(t),$$
$$V_3(t,x,\dot{x})=\int_{t-d_h(t)}^{t}\eta_3^T(y)Q\eta_3(y)dy,$$
$$V_4(t,x,\dot{x})=\int_{t-d_h(t)}^{t}\eta_4^T(y)Z\eta_4(y)dy,$$
$$V_5(t,x,\dot{x})=\int_{t-h_d}^{t}\int_{y}^{t}\dot{x}^T(w)R\dot{x}(w)dw\,dy$$
$$\eta_1(t)\triangleq[x^T(t) \quad {x_d}^T(t)]^T$$
$$\eta_2(t)\triangleq\left[x^T(t-h_d) \quad \int_{t-d_h(t)}^{t}x^T(w)dw \quad \int_{t-h_d}^{t-d_h(t)}x^T(w)dw\right]^T$$
$$\eta_3(y)\triangleq\left[x^T(y) \quad \dot{x}^T(y) \quad \int_{y}^{t}\dot{x}^T(w)dw\right]^T,$$
$$\eta_4(y)\triangleq\left[x^T(y) \quad \dot{x}^T(y) \quad \int_{y}^{t-d_h(t)}\dot{x}^T(w)dw\right]^T$$

According to Schur complement and positive definiteness of S, Q, Z, R, the proposed L-K-F is positive definite from LMI (9)

Define:

$$\xi^T(t)\triangleq[x^T(t) \quad {x_d}^T(t) \quad x^T(t-h_d) \quad \dot{x}^T(t) \quad \dot{x}_d^T(t) \quad \dot{x}^T(t-h_d) \quad \frac{1}{d_h(t)}\int_{t-h_d}^{t-d_h(t)}x^T(w)dw \quad \frac{1}{h_d-d_h(t)}\int_{t-h_d}^{t-d_h(t)}x^T(w)dw\,]$$

Calculating the time derivative of V_i gives:

$$\frac{\partial v_i}{\partial t}=\frac{\partial v_{1i}}{\partial t}+\sum_{v=2}^{5}\frac{\partial v_v}{\partial t}$$

where

$$\frac{\partial v_{1i}}{\partial t}=\xi^T(t)\ Sym\left\{E_1P_iE_2^T\right\}\xi(t),$$
$$\frac{\partial v_2}{\partial t}=\xi^T(t)\ Sym\left\{E_3SE_4^T\right\}\xi(t),\ \frac{\partial v_3}{\partial t}=\xi^T(t)\Omega_1\xi(t),$$
$$\frac{\partial v_4}{\partial t}=\xi^T(t)\Omega_2\xi(t),\ \frac{\partial v_5}{\partial t}=\xi^T(t)h_de_4Re_4^T\xi(t)-\int_{t-h_d}^{t-d_h(t)}\dot{x}^T(w)R\dot{x}(w)dw-\int_{t-d_h(t)}^{t}\dot{x}^T(w)R\dot{x}(w)dw$$

By applying Lemma (1) with LMI (8), the UB of the derivative can be estimated as:

$$\frac{\partial v_i}{\partial t}\leq\xi^T(t)\psi_i\xi(t) \tag{14}$$

According to Lemma (2), it is clear that if LMI (10) holds, then inequality (14) is satisfied.

For $i\in I_1$ consider the L-K-F as bellow:

$$\bar{V}_i(t,x,\dot{x})=\bar{V}_{1i}(t,x)+\bar{V}_2(t,x)+\bar{V}_3(t,x,\dot{x})+\bar{V}_4(t,x,\dot{x})+\bar{V}_5(t,x,\dot{x})$$
$$\bar{V}_{1i}(t,x)=\bar{\eta}_1^T(t)\bar{P}_i\bar{\eta}_1(t),\ \bar{V}_2(t,x)=\bar{\eta}_2^T(t)\bar{S}\bar{\eta}_2(t),$$
$$\bar{V}_3(t,x,\dot{x})=\int_{t-d_h(t)}^{t}\bar{\eta}_3^T(y)\bar{Q}\bar{\eta}_3(y)dy,$$
$$\bar{V}_4(t,x,\dot{x})=\int_{t-d_h(t)}^{t}\bar{\eta}_4^T(y)\bar{Z}\bar{\eta}_4(y)dy,$$
$$\bar{V}_5(t,x,\dot{x})=\int_{t-h_d}^{t}\int_{y}^{t}\dot{\bar{x}}^T(w)\bar{R}\dot{\bar{x}}(w)dw\,dy$$
$$\bar{\eta}_1(t)=[\bar{x}^T(t) \quad \bar{x}_d^T(t)]^T,$$
$$\bar{\eta}_2(t)=\left[\bar{x}^T(t-h_d) \quad \int_{t-d_h(t)}^{t}\bar{x}^T(w)dw \quad \int_{t-h_d}^{t-d_h(t)}\bar{x}^T(w)dw\right]^T$$
$$\bar{\eta}_3(y)=\left[\bar{x}^T(y) \quad \dot{\bar{x}}^T(y) \quad \int_{y}^{t}\dot{\bar{x}}^T(w)dw\right]^T$$
$$\bar{\eta}_4(y)=\left[\bar{x}^t(y) \quad \dot{\bar{x}}^T(y) \quad \int_{y}^{t-d_h(t)}\dot{\bar{x}}^T(w)dw\right]^T$$

A similar approach to $\forall i\in I_0$ can be repeated for this case.

According to LMI (12) and positive definiteness of $\bar{R}$, $\bar{S}$, $\bar{Q}$, $\bar{Z}$, the Lyapunov functional is positive definite. LMI (13) with respect to LMI (11) guarantees that the derivative of Lyapunov functional decreases over time.

Note that $V_2=\overline{V}_2$, $V_3=\overline{V}_3$, $V_4=\overline{V}_4$, $V_5=\overline{V}_5$, and the condition that guarantees the continuity of V_1 and $\overline{V}_1$ at the boundaries can be obtained by using the appropriate continuity matrices.□

The suggested Lyapunov functional is continuous, but its derivative is not continuous at all points of the state space, so Theorem 1 is confirmed if sliding behavior does not occur at the boundaries. It is necessary to study of the appearance of the charming sliding modes at the boundaries for the PWA systems. An analysis to recognize this sliding mode according to the Filippov solution is discussed in [23].

IV. NUMERICAL EXAMPLES

This section, presents three numerical examples to illustrate the effectiveness of the proposed stability conditions. In the first example, it is considered a switched linear time-delay system. In this system, the delay is constant and switching law depends on the current states. In the second example, it is considered the equation of the water level changes in a tank. In this example, the delay is time-varying and switching law depends on the current states. The third example is about the equation of motion of a simple pendulum. In this example, the delay is constant and switching law depends on the delayed states as well as the current states. The characteristics of these examples are given in Table 1. The conservatism of delay-dependent stability conditions is checked by computing the maximum allowable UB of time delay.

Table I
The characteristics of the studied systems

	Type of delay	Dependence type of switching law
Example 1	Constant	Current states
Example 2	Time-varying	Current states
Example 3	Constant	Current states and delayed states

Example 1: Suppose the switched linear time-delay system $\dot{x}=A_i x(t)+A_{di}x(t-d_h)$ with the system matrices given by:

$$A_1=\begin{bmatrix}-0.1 & 0\\ 0 & -0.1\end{bmatrix},$$

$$A_2=\begin{bmatrix}-0.1 & 0\\ 0 & -0.1\end{bmatrix}, A_3=\begin{bmatrix}-0.1 & 0\\ 0 & -0.1\end{bmatrix}, A_4=\begin{bmatrix}-0.1 & 0\\ 0 & -0.1\end{bmatrix};$$

$$A_1^d=\begin{bmatrix}0 & 5\\ -1 & 0\end{bmatrix}, A_2^d=\begin{bmatrix}0 & 1\\ -5 & 0\end{bmatrix} A_3^d=\begin{bmatrix}0 & 5\\ -1 & 0\end{bmatrix} A_4^d=\begin{bmatrix}0 & 1\\ -5 & 0\end{bmatrix};$$

and the cell partition:

$$E_1=\begin{bmatrix}-1 & 1\\ -1 & -1\end{bmatrix}, E_2=\begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix}, E_3=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}, E_4=\begin{bmatrix}1 & -1\\ -1 & -1\end{bmatrix};$$

In this system, the time-delay is constant (d_h), and the switching law depends on the current states. Table 2 shows the maximum allowable time delay obtained by Theorem 1 (h_d) and existing methods [17-21]. As shown in this table, the results obtained by Theorem 1 are less conservative than the currently existing ones.

Table II
Comparison of UB of delay for example 1

Methods	Feasible UB for delay (h_d)
[17]	0.0142
[18,19]	0.0142
[20,21]	0.0168
Theorem 1	0.0193

The state trajectories of the system with initial conditions $[-2,0]^T$ are shown in Fig.1. As shown in this figure, the system is stable for d_h = 0.02 and unstable for d_h = 0.021. According to this, the theoretical maximum UB for time-delay in the system is between 0.02 and 0.021. Therefore, the simulation results confirm less conservatism in Theorem 1.

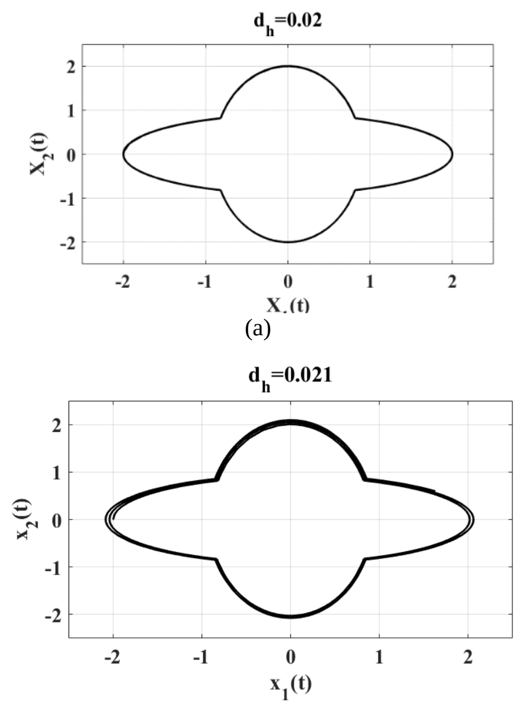

Fig.1: The state trajectories of the system in Example 1 with (a) d_h = 0.02 and (b) d_h = 0.021

Example 2: Fig.2 shows a water tank and a pipe with length of *L*. In this example, a water tank system with a nonlinear model has the following statement [18]:

$$\dot{x}(t)=\frac{1}{A\varphi}\left(-\frac{1}{H}\sqrt{\varphi g x(t)}+u_{in}\left(t-d_h(t)\right)\right) \quad (15)$$

where $g=9.8\ ms^{-2}$, $\varphi=1000\ kgm^{-3}$, $H=11.3882\ m^{-1/2}kg^{-1/2}$ and $A=10\ m^2$. The pipe length L makes a time-varying delay ($d_h(t)$) in the water inflow to the tank. The goal is to keep the water level at x=0.5cm. We obtained a PWA model of the system with linearization around two operating point $x_0=0.25$ cm, $x_0=0.75$ cm, as follow as:

$$\dot{x}(t)=\frac{-gx(t)}{2AH\sqrt{0.25\varphi g}}+\frac{1}{A\varphi}u_{in}(t-d(t))+\left(\frac{0.25g}{2AH\sqrt{0.25\varphi g}}-\frac{\sqrt{0.25\varphi g}}{\varphi AH}\right);\quad 0\leq x(t)<0.5$$

$$\dot{x}(t)=\frac{-gx(t)}{2AH\sqrt{0.75\varphi g}}+\frac{1}{A\varphi}u_{in}(t-d(t))+\left(\frac{0.75g}{2AH\sqrt{0.75\varphi g}}-\frac{\sqrt{0.75\varphi g}}{\varphi AH}\right);\quad 0.5\leq x(t)<1$$

and the cell partition is obtained:

$$E_1=\begin{bmatrix}-1 & 0.5\\ 0 & 1\end{bmatrix},\ E_2=\begin{bmatrix}1 & -0.5\\ 0 & 1\end{bmatrix}$$

The switching law is based on the current states.

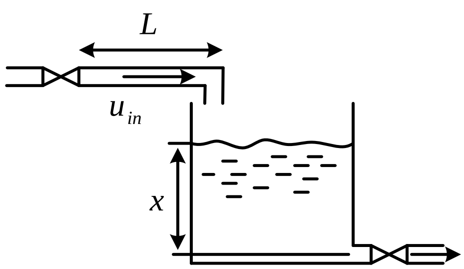

Fig.2: Water tank

Suppose a control input denoted as below:

$u_{in}(t)=0.2A\varphi x(t),\qquad 0\leq x(t)<0.5$

$u_{in}(t)=-0.1A\varphi x(t),\qquad 0.5\leq x(t)<1$

Fig.3 shows the level of water for the system without delay. As shown in this figure, the system without delay is stable. But in practice the system should be considered with delay. Incrasing the delay can cause closed-loop instability. Therefore, it is necessary to found the maximum allowable upper bound of the time-delay for which the stability of the closed loop system is guaranteed.

Table 3 shows the results for the maximum allowable time-varying delay for various bounds of the time derivative of the time delay obtained by Theorem 1. In Table 3, μ is the UB of the derivative of time delay, as shown in (2). It can be verified with the simulation that the nonlinear system is marginal stable for constant delay $d_h=6.53$ as shown in Fig. 4 and the system is unstable for $d_h=6.54$. The maximum allowable constant delay obtained with Theorem 1 for the equivalence PWA system is $h_d=6.265$.

Table III
The maximum allowable time-varying delay for several of μ

μ	0.1	0.3	0.6
Maximum allowable delay	6.09	4.94	0.02

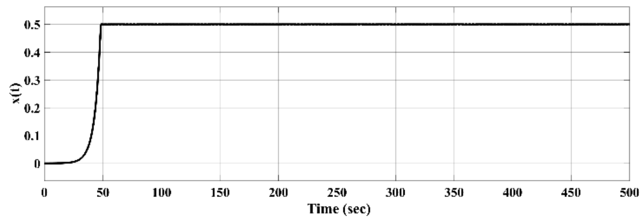

Fig. 3: The water level in the tank without delay

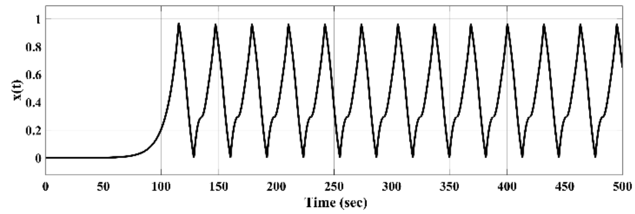

Fig. 4: The water level in the tank with constant input delay $d_h=6.53$

Example 3: In this example, a simple pendulum motion model with constant time delay (d_h) between the sensor and the controller is considered as bellow [20]:

$$M_pL_p^2\ddot{\vartheta}(t)=-M_pgL_p\sin(\vartheta(t))+T(t-d_h)\qquad(16)$$

where M_p=1kg is the mass of pendulum, L_p=9.8m is the length of pendulum, g is the gravity of earth, and T is the input torque. A PWA model of this system with time delay and the switching conditions, which depend on both the states and the delayed states, is obtained in [20]:

$$\begin{bmatrix}\dot{\vartheta}(t)\\ \ddot{\vartheta}(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -0.825 & 0\end{bmatrix}\begin{bmatrix}\vartheta(t)\\ \dot{\vartheta}(t)\end{bmatrix}+\begin{bmatrix}0 & 0\\ -1 & -1\end{bmatrix}\begin{bmatrix}\vartheta(t-d_h)\\ \dot{\vartheta}(t-d_h)\end{bmatrix};\ \forall\begin{bmatrix}\vartheta(t-d_h)\\ \dot{\vartheta}(t-d_h)\end{bmatrix}\in\Psi 1$$

$$\begin{bmatrix}\dot{\vartheta}(t)\\ \ddot{\vartheta}(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -0.825 & 0\end{bmatrix}\begin{bmatrix}\vartheta(t)\\ \dot{\vartheta}(t)\end{bmatrix}+\begin{bmatrix}0 & 0\\ -5 & -5\end{bmatrix}\begin{bmatrix}\vartheta(t-d_h)\\ \dot{\vartheta}(t-d_h)\end{bmatrix};\ \forall\begin{bmatrix}\vartheta(t-d_h)\\ \dot{\vartheta}(t-d_h)\end{bmatrix}\in\Psi 2$$

$$\begin{bmatrix}\dot{\vartheta}(t)\\ \ddot{\vartheta}(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -0.4875 & 0\end{bmatrix}\begin{bmatrix}\vartheta(t)\\ \dot{\vartheta}(t)\end{bmatrix}+\begin{bmatrix}0 & 0\\ -1 & -1\end{bmatrix}\begin{bmatrix}\vartheta(t-d_h)\\ \dot{\vartheta}(t-d_h)\end{bmatrix};\ \forall\begin{bmatrix}\vartheta(t-d_h)\\ \dot{\vartheta}(t-d_h)\end{bmatrix}\in\Psi 3$$

$$\begin{bmatrix}\dot{\vartheta}(t)\\ \ddot{\vartheta}(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -0.4875 & 0\end{bmatrix}\begin{bmatrix}\vartheta(t)\\ \dot{\vartheta}(t)\end{bmatrix}+\begin{bmatrix}0 & 0\\ -5 & -5\end{bmatrix}\begin{bmatrix}\vartheta(t-d_h)\\ \dot{\vartheta}(t-d_h)\end{bmatrix};\ \forall\begin{bmatrix}\vartheta(t-d_h)\\ \dot{\vartheta}(t-d_h)\end{bmatrix}\in\Psi 4$$

with partitions:

$\Psi 1:\{-0.7854\leq\vartheta(t)\leq 0,\ -0.7854\leq\vartheta(t-d_h)\leq 0\}$;

$\Psi 2:\{-0.7854\leq\vartheta(t)\leq 0,\ 0\leq\vartheta(t-d_h)\leq 0.7854\}$;

$\Psi 3:\{0\leq\vartheta(t)\leq 0.7854,\ -0.7854\leq\vartheta(t-d_h)\leq 0\}$;

$\Psi 4:\{0\leq\vartheta(t)\leq 0.7854,\ 0\leq\vartheta(t-d_h)\leq 0.7854\}$;

The simulation results are given in Fig. 5, and 6 and show that the system is stable with $d_h=0.19$, but becomes unstable when $d_h=0.196$, which implies that the theoretical UB of the delay is between 0.19 and 0.196. The maximum allowable delay obtained with Theorem 1 for this system is $h_d=0.18$. Due to switching based on delayed-states, the methods proposed in

[17-19] cannot deal with this example and the maximum upper bound that is obtained in [20], is h_d=0.161. As a result, the proposed approach can achieve less conservative estimate of the UB for the delay in this system.

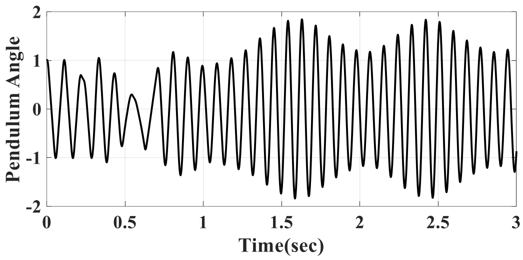

Fig. 5: The pendulum angle for d_h=0.19

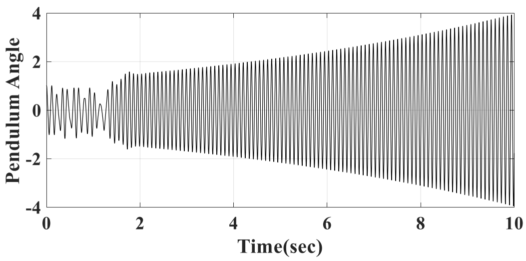

Fig. 6: The pendulum angle for d_h=0.196

V. Conclusions

This paper introduced novel stability conditions for the PWA-TVD system. It is considered that the system switches are based on both the states and the delayed states. The stability criteria for the PWA-TVD system have been established in terms of LMIs. By choosing the improved L-K-F and using new inequality for the estimation of its derivative, the derived results are less complex and conservative. Future research can focus on several extensions such as stability condition for uncertain PWA systems, the use of different Lyapunov functional to obtain less conservative results, and the use of the presented theorem to stabilize PWA systems with time delay and uncertainty.

Appendix

The following algorithm is used to find the maximum allowable upper bound for delay:

Step1: Select the positive constants h_d and μ.

Step2: Solve LMIs (8)-(13) and then check the feasibility of LMIs for $\forall d_h(t)\in\{0,h_d\}$, $\dot{d}_h(t)\in\{-\mu,\mu\}$.

Step3: If LMIs will be feasible, increase h_d ,and then solve LMIs (8)-(13) again (step2), else if LMIs will be infeasible, exit the algorithm.

References

[1] M. S. Branicky, "Studies in hybrid systems: Modeling, analysis, and control," Sc.D. dissertation, Dept. of Electrical Engineering and Computer Science, MIT, Cambridge, MA, June 1995.

[2] J. P. Richard, "Time-delay systems: An overview of some recent advances and open problems," *Automatica*, vol. 39, no. 10, pp. 1667–1694, 2003.

[3] H. Ye, A. N. Michel, and L. Hou, "Stability theory for hybrid dynamical systems," *in Proc. IEEE Decision and Control Conference*, New Orleans, LA, Dec. 1995.

[4] M. Johansson and A. Rantzer, "*Computation of piecewise quadratic Lyapunov functions for hybrid systems*," Dept. Automatic Control, Lund Institute of Technology, Tech Rep. TFRT-7549, June 1996.

[5] M. S. Branicky, "Multiple lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Transaction on Automatic Control*, vol. 43, no. 4, pp. 475–482, April 1998.

[6] A. Hassibi, and S. Boyd, "Quadratic stabilization and control of piecewise-linear systems," *Proceedings of the American Control Conference, Philadelphia*, PA, vol. 6, pp. 3659–3664, June 1998.

[7] A. Bemporad, G. Ferrari-Trecate, and M. Morari, "Observability and controllability of piecewise affine and hybrid systems," *IEEE Transaction on Automatic Control*, vol. 45, no. 10, pp. 1864–1876, Oct. 2000.

[8] K. Gu, V. L. Kharitonov, and J. Chen, "Stability of Time-Delay Systems, " *Springer Science & Business Media*, 2003.

[9] H. B. Zeng, Y. He, M. Wu, and S. P. Xiao, "Less conservative results on stability for linear systems with a time-varying delay," *Optimal control, application and Method*., vol. 34, no. 6, pp. 670–679, 2013.

[10] W. kwon, B. Koo, and S. M. Lee, "Novel Lyapunov–Krasovskii functional with delay-dependent matrix for stability of time-varying delay systems", *Applied Mathematics and Computation*, vol. 320, pp.149-157, 2018.

[11] F. Long, L. Jiang, Y. He, and M. Wu, "Stability analysis of systems with time-varying delay via novel augmented Lyapunov–Krasovskii functionals and an improved integral inequality", *Applied Mathematics and Computation*, vol. 357, pp.325-337, 2019.

[12] T. H. Lee, and J. H. Park, "Improved stability conditions of time-varying delay systems based on new Lyapunov functionals", *Journal of the Franklin Institute*, vol. 355, no.3, pp.1176-1191, 2018.

[13] A. Seuret and F. Gouaisbaut, "Wirtinger-based integral inequality: Application to time-delay systems," *Automatica*, vol. 49, no. 9, pp. 2860–2866, 2013.

[14] H. Zeng, Y. He, M. Wu, and J. She, "Free-Matrix-Based integral inequality for stability analysis of systems with time-varying delay," *IEEE Transaction on Automatic Control*, vol. 60, no. 10, pp. 2768–2772, 2015.

[15] R. Datta, R. Dey, B. Bhattacharya, R. Saravanakumar, and C. K. Ahn, "New double integral inequality with application to stability analysis for linear retarded systems", *IET Control Theory & Applications*, vol. 13, no. 10, pp.1514-1524,2019.

[16] R. Zhang, D. Zeng, J. H. Park, S. Zhong, Y. Liu, and X. Zhou, "New approaches to stability analysis for time-varying delay systems", *Journal of the Franklin Institute*, vol. 356, no.7, pp. 4174-4189, 2019.

[17] V. Kulkarni, M. Jun, and J. Hespanha, "Piecewise quadratic Lyapunov functions for piecewise affine time-delay systems," *Proceedings of the American Control Conference*, vol. 5, no.30, pp. 3885- 3889, 2004.

[18] K. Moezzi, L. Rodrigues, and A. G Aghdam, "Stability of Uncertain Piecewise Affine Systems with Time Delay:

Delay-Dependent Lyapunov Approach," *International Journal of Control*, vol. 82, no. 8, pp. 1423-1434, 2009.
[19] K. Moezzi, L. Rodrigues, and A. G. Aghdam, "Stability of uncertain piecewise affine systems with time-delay," American Control Conference, pp. 2373-2378, 2009.
[20] S. Duan, J. Ni, and A. G. Ulsoy, "An improved LMI-based approach for stability of piecewise affine time-delay systems with uncertainty," *International Journal of Control*, vol. 85, no. 9, pp. 1218-1234, 2012.
[21] S. Duan, J. Ni, and A. G. Ulsoy, "Stability criteria for uncertain piecewise affine time-delay systems," *American Control Conference (ACC)*, pp. 5460-5465, 2012.
[22] C. Fiter and E. Fridman, "Stability of piecewise affine systems with state-dependent delay, and application to congestion control," *in 52nd IEEE conference on decision and control*, pp. 1572-1577, 2013.
[23] M. Johansson, "Piecewise Linear Control Systems," *Springer-Verlag*, 2003.
[24] T. H. Lee, H. Ju, and S. Xu, "Relaxed conditions for stability of time-varying delay systems," *Automatica*, vol. 75, pp. 11-15, 2017.
[25] R. E. Skelton, T. Iwasaki, and K. M. Grigoradis, "A unified algebraic approach to linear control," *design. New York: Taylor and Francis*, 1997.

Salehe Afsharian received the B.S. and M.S. degree in control engineering from the Ferdowsi University of Mashhad (FUM), Iran, in 2009 and 2012, respectively. Since 2014, she is doing a PhD dissertation in Control Engineering at Department of Electrical and Computer Engineering, University of Birjand, Iran. Her research interests include Hybrid System, Time-delay Systems, Stability Analysis, Model Predictive Control, Modeling and Simulation, Renewable and Sustainable Energy, and Control.

Hussein Eliasi received the B.S. degree in electrical engineering from the Ferdowsi University of Mashhad (FUM), Iran, in 2001. He received the M.S. and the Ph.D. degrees from Amirkabir University of technology (Tehran), Iran, in 2005 and 2011, respectively. He is currently an assistance professor in faculty of electrical engineering at University of Birjand, Iran. His research interests include Constrained Nonlinear Dynamics Systems, Robust Control, Parameter Optimization of Model Predictive Control, Transient Stability in Power Systems and Load Following by Nuclear Power Plants.