A Semi-Analytic Method for Solving a Class of Non-Linear Optimal Control Problems

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This paper, proposes an approximate analytical method to solve a class of optimal control problems. This method is an enhancement of the variational iteration method (VIM) that named modified variational iteration method (MVIM) and eliminates all additional calculations in VIM, thus requires less time to do the calculations. In this approach, first, the optimal control problem is converted into a nonlinear two-point boundary value problem via the Pontryagins maximum principle, and then we applied the MVIM method to solve this boundary value problem. This suggested method is suitable for a large class of nonlinear optimal control problems that for the non-linear part of the problem, we used the Taylor series expansion. In the end, three examples are provided to demonstrate the simplicity and efficiency of the method. Numerical results of the proposed method versus other methods is presented in tables. All calculations were carried out using Mathematica software.

Article Info

I.   INTRODUCTION

Recently, non-linear optimal control problems (NOCPs) have been applied in different fields such as aircraft systems [1, 2], biomedicine [3] and robotics [4]. Because of the importance of this type of problem and its impact on science and engineering, researchers have shown considerable interest in this issue. Indeed, numerical approaches are usually applied to solve the problems In direct methods, the problem can be converted into a linear or non-linear programming by using the discretization or parametrization techniques [5]. Indirect methods, on the other hand, are based on the Pontryagin's maximum principle [6]. Various kinds of techniques have been proposed to solve NOCP's. Shirazian et al. [7] suggested the application of the variational iteration method along with a shooting method in order to solve the extreme conditions resulting from the Pontryagin's maximu principle. Kafash et al. [8] offered a numerical approach for

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solving optimal control problems (OCP) using the Boubaker polynomials expansion scheme. This approach is based on state parametrization. Indeed, the state variable is approximated by Boubaker polynomials with unknown coefficients and then performance index and boundary conditions are transformed into some algebraic equations. A new analytic technique based on VIM and some modifications was suggested to solve NOCPs in [9]. Jafari et al. [10] has been shown that the method proposed in [9] is exactly the same iterative formula as the ADM and HPM for solving NOCPs. Alipour et al. [11] introduced an approach for NOCPs which makes use of homotopy analysis and parametrization methods. Actually an appropriate parametrization of control is applied and state variables are computed using homotopy analysis method. Jajarmi et al. [12] came up with a novel analytical technique, called OHPM, to solve a class of NOCPs. In this paper, the authors argue that the proposed algorithm method has low computational complexity and fast convergence rate. A numerical technique based on the linear B-spline polynomials offered to solve OCP [13]. In this process, state and control functions are approximated in terms of B-spline functions. Also, in [14], OCPs were solved through the spectral homotopy analysis method (SHAM). SHAM is combination of

the hybrid spectral collocation technique and the homotopy analysis method. This article points out that SHAM is stronger than HAM due to it removes restrictions of the HAM such as the requirement for the solution to conform to the so-called rule of solution expression and the rule of coefficient ergodicity. In addition, more numerical methods in this area can be found in [15, 16, 17, 18, 19].

Ji-Huan He adevised a method to solve non-linear differential equations using VIM [20, 21]. VIM also features a number of disadvantages which decrease its power. These are mainly associated with repetitive calculations and the introduction of excessive unnecessary terms. To overcome these downsides, Abassy et al. [22] proposed the modified variational iteration method (MVIM). Besides, MVIM improved the rate of convergence. This paper uses MVIM as a semi-analytical method to solve NOCPs. The proposed method is an indirect method and does not require discretization, linearization or transformation. In this approach, first, the optimal control problem is converted into a nonlinear two-point boundary value problem via the Pontryagin's maximum principle, and then we applied the MVIM method to solve this boundary value problem. The examples show that our proposed method is rewarding thanks to its simplicity and small computation costs. The paper is organized as follows: Section 2, introduces NOCPs; MVIM for NOCPs is discussed in Section 3; Section4 simulates the numerical examples to check the efficiency of the method; and Section 5 draws a number of conclusions based on the results.

II. NONLINEAR OPTIMAL CONTROL PROBLEMS

Consider the following nonlinear dynamical system:

$$x'(t) = f(t, x(t)) + g(t, x(t))u(t), t \in [t_0, t_f] \tag{1}$$
$$x(t_0) = x_0 , \ x(t_f) = x_f,$$

Where $x(t) \in R^n$ denotes the state variable, $u(t) \in R^m$ is the control variable, and x_0 and x_f are the given initial and final states at t_0 and t_f respectively. Moreover, $f(t, x(t)) \in R^n$ and $g(t, x(t))u(t) \in R^{n \times m}$ are two continuously differentiable functions in all arguments. Our purpose is to minimize the quadratic objective function.

$$J[x, u] = \frac{1}{2} \int_{t_0}^{t_f} (x^T(t)Qx(t) + u^T(t)R\,u(t)\,)\,dt \tag{2}$$

subject to the non-linear system (1), where $Q \in R^{n \times n}$ and $R \in R^{m \times m}$ are positive semi-definite and positive definite matrices, respectively. Since the performance index (2) is convex, the following extreme necessary conditions are also sufficient for optimality:

$$x' = f(t, x) + g(t, x)u^*$$
$$\lambda' = -H_x(x, u^*, \lambda),$$
$$u^* = argmin_u \ H(x, u, \lambda) \tag{3}$$
$$x(t_0) = x_0 , \ x(t_f) = x_f,$$

where

$$H(x, u, \lambda) = \frac{1}{2}[x^T Q x + u^T R u] + \lambda^T \ [f(t, x) + g(t, x)u]$$

is the Hamiltonian function. Similarly (3) can be written in the form of:

$$x' = f(t, x) + g(t, x)[-R^{-1} \ g^T(t, x)\lambda],$$
$$\lambda' = -(Qx + \left(\frac{\partial f(t,x)}{\partial x}\right)^T \lambda +$$
$$\sum_{i=1}^{n} \lambda_i[-R^{-1} \ g^T(t, x)\lambda]^T \frac{\partial g_i(t,x)}{\partial x})$$
$$x(t_0) = x_0 , \ x(t_f) = x_f \tag{4}$$

where , $\lambda(t) \in R^n$ is the co-state vector with the ith component $\lambda_i(t), i = 1, \dots, n,$ and $g(t, x) = [g_1(t, x), \dots, g_n(t, x)]^T$ with $g_i(t, x) \in R^n$, $i = 1, \dots, n.$ Also, the optimal control law is obtained by

$$u^* = -R^{-1} \ g^T(t, x)\lambda \tag{5}$$

There is no analytical solution for solving such a two-point boundary-value problem (TPBVP) in (4). Therefore, it is highly recommended to calculate analytic approximate or numerical solutions for it. We shall apply MVIM to solve the following initial value problem. Taylor series expansion is used here for the non-linear part of the problem.

III. THE MODIFIED VARIATIONA ITERATION METHOD

In this section, consider the following differential equations,

$$LV(t) + RV(t) + NV(t) = g(t)$$
$$V(0) = f(t), \tag{6}$$

where L = $\frac{d}{dt}$, R is a linear operator, N is a non-linear term and $g(t)$ is an inhomogeneous term. Using VIM [20, 21] to solve the non-linear differential equation (6), the following variational iteration formula can be obtained:

$$V_{n+1}(t) = V_n(t) - \int_0^t \{L(V_n(\tau)) + R(V_n(\tau)) + N(V_n(\tau)) - g(\tau)\}\,d\tau. \tag{7}$$

It has been shown [22] that equation (7) is equivalent to the following equation:

$$V_{n+1}(t) = V_n(t) - \int_0^t \{R(V_n(\tau) - V_{n-1}(\tau)) + (G_n(\tau) - G_{n-1}(\tau)) - g(\tau)\}\,d\tau, \tag{8}$$

where $V_{-1} = 0, V_0 = f(t), V_1 = V_0 - \int_0^t \{R(V_0 - V_{-1}) + (G_0 - G_{-1}) - g\}d\tau$ and $G_n(t)$ is obtained from $NV_n(t) = G_n(t) + O(t^{n+1})$. The Maclaurin series expansion is employed here for the non-linear part of the problem. Eq. (8) can be solved iteratively to obtain an approximate solution that takes the form $V(t) \simeq V_n(t)$, where n is the final iteration step.

Theorem 1. Suppose that $V_0(t) = V_0$ and the iterative sequence $\{V_n(t)\}$ obtained from (7) converges to $V(t)$; then $V(t)$ is the solution of Eq. (6).

Proof. Considering the limits in the iterative formula (7), it follows that

$$\lim_{n \to \infty} V_{n+1} = \lim_{n \to \infty} V_n - \int_0^t \lim_{n \to \infty} [L(V_n(\tau)) + R(V_n(\tau)) + N(V_n(\tau))]\,d\tau.$$

By considering $\lim_{n \to \infty} V_n = V$ and the continuity of N operator, we will have

$$\int_0^t [L(V(\tau)) + R(V(\tau)) + N(V(\tau))]d\tau = 0.$$

Then, differentiation of both sides concerning t yields

$$LV(t) + RV(t) + NV(t) = 0. \tag{9}$$

Clearly, $V(t)$ satisfies (6). Moreover, if $t = 0$, then form (7), $V_{n+1}(0) = V_n(0)$, for every $n \geq 0$. Thus $V_0(0) = V_n(0) = V_0$. Hence, $V(t)$ is the solution of Eq. (6) and the proof is complete. Since the Maclaurin series is convergent, equation (8) also converges.

The Modified Variational Iteration Method for Nocp's

We consider equation (4) as follows:
$x'(t) + L\,x(t) + Nx(t) = 0$,
$\lambda'(t) + L\,\lambda(t) + N\,\lambda(t) = 0$,
$x(t_0) = x_0,\ x(t_f) = x_f$, $\qquad$ (10)

where L is a linear operator and N is a non-linear operator. To solve system (10) with MVIM, we should answer the following system:
$x'(t) + L\,x(t) + Nx(t) = 0$,
$\lambda'(t) + L\,\lambda(t) + N\,\lambda(t) = 0$,
$x(t_0) = x_0,\ \lambda(t_0) = \alpha$. $\qquad$ (11)

In equation (11)

$$Lx(t) + Nx(t) = -(f(t,x) + g(t,x)[-R^{-1}\,g^T(t,x)\lambda]),$$

$$L\lambda(t) + N\lambda(t) = \left(Qx + \left(\frac{\partial f(t,x)}{\partial x}\right)^T \lambda + \right.$$

$$\left. \sum_{i=1}^n \lambda_i [-R^{-1}\,g^T(t,x)\lambda]^T \frac{\partial g_i(t,x)}{\partial x}\right). \tag{12}$$

To solve equation (11) with MVIM, we construct the below iterations formula according to equation (8):

$$x_{n+1}(t) = x_n(t) - \int_0^t \{R(x_n(\tau) - x_{n-1}(\tau)) + (G_n(\tau) -$$

$$G_{n-1}(\tau))\}\,d\tau, \tag{13}$$

$$\lambda_{n+1}(t) = \lambda_n(t) - \int_0^t \{R(\lambda_n(\tau) - \lambda_{n-1}(\tau)) + (K_n(\tau) -$$

$$K_{n-1}(\tau))\}\,d\tau, \tag{14}$$

where $x_{-1}(t) = 0,\ \lambda_{-1}(t) = 0, x(0) = x_0$ and $\lambda(0) = \alpha$. We have:

$$x_1(t) = x_0(t) - \int_0^t \{R(x_0(\tau) - x_{-1}(\tau)) + (G_0(\tau) -$$

$$G_{-1}(\tau))\}\,d\tau,$$

$$\lambda_1(t) = \lambda_0(t) - \int_0^t \{R(\lambda_0(\tau) - \lambda_{-1}(\tau)) + (K_0(\tau) -$$

$$K_{-1}(\tau))\}\,d\tau,$$

and $G_n(t)\ and\ K_n(t)$ are obtained from
$Nx_n(t) = G_n(t) + O(t^{n+1})$ and $N\lambda_n(t) = k_n(t) + O(t^{n+1})$. Eqs. (13) and (14) can be solved iteratively to obtain an approximate solution that takes the form $x(t) \simeq x_n(t)$ and $\lambda(t) \simeq \lambda_n(t)$, where n is the final iteration step. The optimal control law is obtained by

$$u^* = -R^{-1}g^T(t,x)\lambda. \tag{15}$$

For stopping criterion, we consider the following criterion: $\left|\frac{J_{n+1} - J_n}{J_{n+1}}\right| < \varepsilon$, where $\varepsilon > 0$ should be chosen according to the desirable accuracy.

Numerically   Example

In this section, we have solved three examples to illustrate the simplicity and efficiency of the proposed method.

Example 1. Consider the following nonlinear optimal control problem

$$minimize J = \int_0^1 u^2(t)dt,$$

Subject to:

$$x' = \frac{1}{2}\,x^2(t)sinx(t) + u(t), \qquad t \in [0,1],$$

$$x(0) = 0, \qquad x(1) = 0.5, \tag{16}$$

The necessary equations for the optimal control are given as:

$$x' = \frac{1}{2}\,x^2(t)sinx(t) - \frac{1}{2}\lambda(t), \qquad t \in [0,1],$$

$$\lambda' = -\lambda(t)x(t)sinx(t) - \frac{1}{2}\lambda(t)x^2(t)cosx(t),$$

$$x(0) = 0, \qquad \lambda(0) = \alpha, \tag{17}$$

that

$$u(t) = -\frac{1}{2}\lambda(t).$$

In this example, applying the following iteration formula

$$x_{n+1} = x_n - \int_0^t \{R(x_n - x_{n-1}) + (G_n - G_{n-1})\}d\tau,$$

$$\lambda_{n+1} = \lambda_n - \int_0^t \{R(\lambda_n - \lambda_{n-1}) + (G_n - G_{n-1})\}d\tau,$$

we consider

$$R\big(x(t)\big) = \frac{1}{2}\lambda(t),$$

$$G\big(x(t)\big) = -\frac{1}{2}\,x^2(t)sinx(t),$$

$$R\big(\lambda(t)\big) = 0,$$

$$G\big(\lambda(t)\big) = \lambda(t)x(t)sinx(t) + \frac{1}{2}\lambda(t)x^2(t)cosx(t), \tag{18}$$

by applying Mathematica software, five-term approximations for x and λ were obtained as follows:

$$x_5(t) = -\frac{t\alpha}{2},$$

$$\lambda_5(t) = \alpha - \frac{t^3\alpha^3}{8} + \frac{t^5\alpha^5}{192}.$$

In this case, we should have

$$x_5(1) = -\frac{\alpha}{2},$$

where α is an unknown parameter which, will be obtained from the final state condition $x(t_f) = x_f$. Here, the value of α is derived from

$$x_5(1) = -\frac{\alpha}{2} = 0.5,$$

that is $\alpha = -1$. The optimal control is as follows:

$$u(t) \simeq u_5(t) = -\frac{1}{2}\lambda_5(t) = \frac{1}{2}\left(1 - \frac{t^3}{8} + \frac{t^5}{192}\right).$$

We consider $\varepsilon = 10^{-3}$. Once the proposed method is applied, the numerical results for J_i and stopping criterion are as given in Table I. The maximum absolute error of the proposed method, modal series method [23], and measure theory method [24] are presented in Table II, which shows the proposed method has achieved similar results with modal series method. In addition, it should be noted that the basic VIM can not be calculated more than two iterations for example above. Also, the obtained numerical solution for $x(t)$ and $u(t)$ in five iterative are depicted in Fig 1.

**Table I**: Numerical results for different iteration, Example 1

i	J_i	$\left|\frac{J_i - J_{i-1}}{J_i}\right|$
1	0.25000	-
2	0.25000	0
3	0.23493	6.4146767×10^{-2}
4	0.23493	0
5	0.23533	1.6997408×10^{-3}

Table II: Numerical results for the proposed method versus other methods, Example 1

Method	Objective value	Max state error	CPU time
Proposed Method (N=5)	0.2353	3.2×10^{-5}	0.01562
Modal series method (n=5)	0.2353	2.8×10^{-5}	-
Measure theory method	0.2425	4.3×10^{-3}	-
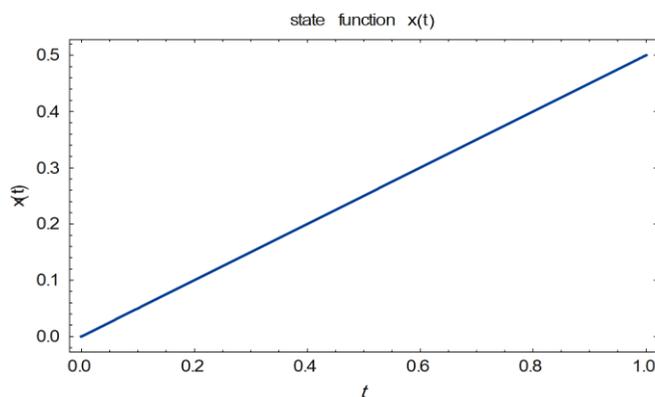
Fig. 1(a): Suboptimal state and control $x(t)$ Example 1.
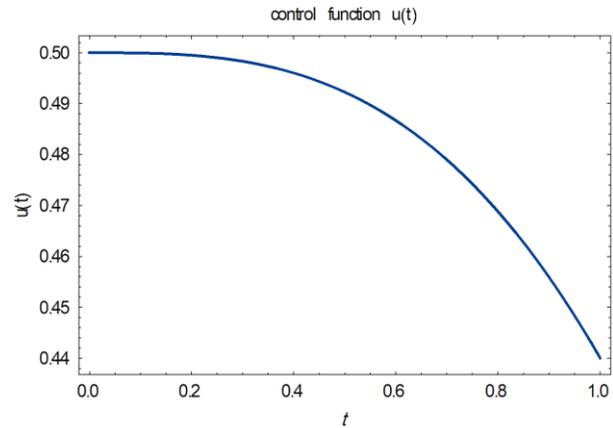
Fig. 1(b): Suboptimal state and control $u(t)$, Example 1.

Example 2. We consider the optimal maneuvers of a rigid asymmetric space craft [2]. The Euler's equations for the angular velocities of the spacecraft are given by:

$$x' = \begin{bmatrix} x'_1(t) \\ x'_2(t) \\ x'_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{I_3 - I_2}{I_1} x_2(t)x_3(t) \\ -\frac{I_1 - I_3}{I_2} x_1(t)x_3(t) \\ -\frac{I_2 - I_1}{I_3} x_1(t)x_2(t) \end{bmatrix} + $$

$$\begin{bmatrix} \frac{1}{I_1} & 0 & 0 \\ 0 & \frac{1}{I_2} & 0 \\ 0 & 0 & \frac{1}{I_3} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix},$$

where x_1, x_2 and x_3 are the angular velocities of the spacecraft, u_1, u_2 and u_3 are the control torques, and $I_1 = 86.24$, $I_2 = 85.07$, and $I_3 = 113.59$ kg m^2 are the spacecraft principle inertia. The optimal control is to find the control $u(t)$ ($t \in [o, T]$) that minimize the cost function

$$J[x, u] = \frac{1}{2}\int_0^{100} (x^T(t)Qx(t) + u^T(t)Ru(t))dt,$$

Where: $Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

In addition, the following boundary conditions should be satisfied:
$x_1(0) = 0.01 \,^r/_s, x_2(0) = 0.005 \,^r/_s, x_3(0) = 0.001 \,^r/_s,$
$x_1(100) = x_2(100) = x_3(100) = 0 \,^r/_s.$
According to the Pontryagin's maximum principle, the following non-linear TPBVP should be solved:

$$x_1'(t) = -\frac{\lambda_1(t)}{I_1^2} - \frac{I_3 - I_2}{I_1}x_2(t)x_3(t),$$

$$x_2'(t) = -\frac{\lambda_2(t)}{I_2^2} - \frac{I_1 - I_3}{I_2}x_1(t)x_3(t),$$

$$x_3'(t) = -\frac{\lambda_3(t)}{I_3^2} - \frac{I_2 - I_1}{I_3}x_1(t)x_2(t),$$

$$\lambda'_1(t) = \frac{I_1 - I_3}{I_2} x_3(t)\lambda_2(t) + \frac{I_2 - I_1}{I_3} x_2(t)\lambda_3(t),$$

$$\lambda'_2(t) = \frac{I_3 - I_2}{I_1} x_3(t)\lambda_1(t) + \frac{I_2 - I_1}{I_3} x_1(t)\lambda_3(t),$$

$$\lambda'_3(t) = \frac{I_3 - I_2}{I_1} x_2(t)\lambda_1(t) + \frac{I_1 - I_3}{I_2} x_1(t)\lambda_2(t),$$

$x_1(0) = 0.01\,{}^r/_s$, $x_2(0) = 0.005\,{}^r/_s$, $x_3(0) = 0.001\,{}^r/_s$,

$x_1(100) = x_2(100) = x_3(100) = 0\,{}^r/_s.$

and the optimal control law is given by:

$$\begin{cases} u_1(t) = -\dfrac{\lambda_1(t)}{I_1} \\[2mm] u_2(t) = -\dfrac{\lambda_2(t)}{I_2} \qquad t \in [0,100], \\[2mm] u_3(t) = -\dfrac{\lambda_3(t)}{I_3} \end{cases}$$

To solve the above problem by means of MVIM, we choose the initial approximations $x_{1,0}(t) = 0.01$, $x_{2,0}(t) = 0.005$, $x_{3,0}(t) = 0.001$, $\lambda_{1,0}(t) = \alpha_1$, $\lambda_{2,0}(t) = \alpha_2$, $\lambda_{3,0}(t) = \alpha_3$ Four-term approximations for $\lambda_1, \lambda_2, \lambda_3, x_1, x_2$ and x_3, were obtained as follows:

$x_{14}(t) = 0.01 - 9.57399 * 10^{-10}\, t^2 - 3.80068 * 10^{-13}\, t^3 - 8.41885 * 10^{-17}\, t^4 - t(1.65353 * 10^{-6} + 0.000134457\alpha_1) + 4.83265 * 10^{-11}\, t^3 \alpha_1 + \cdots$

$x_{24}(t) = 0.005 + 5.62075 * 10^{-10}\, t^2 + 6.97602 * 10^{-14}\, t^3 - 9.74227 * 10^{-17}\, t^4 - 4.44624 * 10^{-8}\, t^2 \alpha_1 - 8.35108 * 10^{-11}\, t^3 \alpha_1 + 1.00806 * 10^{-14}\, t^4 \alpha_1 + \cdots$

$x_{34}(t) = 0.001 + 2.45993 * 10^{-10}\, t^2 - 1.59378 * 10^{-17}\, t^4 - 7.10014 * 10^{-8}\, t^2 \alpha_1 + 2.73532 * 10^{-27}\, t^3 \alpha_1 - 2.77608 * 10^{-15}\, t^4 \alpha_1 + 1.80313 * 10^{-14}\, t^4 \alpha_1{}^2 + \cdots,$

$\lambda_{14}(t) = \alpha_1 - 9.57399 * 10^{-8}\, t^2\alpha_1 - 3.65045 * 10^{-11}\, t^3 \alpha_1 - 1.08007 * 10^{-14}\, t^4 \alpha_1 + 2.61537 * 10^{-12}\, t^4 \alpha_1{}^2 - 2.92101 * 10^{-12}\, t^3 \alpha_2 + 4.63115 * 10^{-15}\, t^4 \alpha_2 + \cdots,$

$\lambda_{24}(t) = 2.59506 * 10^{-11}\, t^3\alpha_1 - 4.76375 * 10^{-15}\, t^4 \alpha_1 - 7.0635 * 10^{-9}\, t^3 \alpha_1{}^2 + 3.50454 * 10^{-15}\, t^4 \alpha_1{}^2 + \alpha_2 + 1.12415 * 10^{-7}\, t^2 \alpha_2 - 3.65045 * 10^{-11}\, t^3 \alpha_2 - 1.02222 * 10^{-14} t^4\alpha_2 + \cdots,$

$\lambda_{34}(t) = -1.05879 * 10^{-22}\, t^2\alpha_1 + 6.77927 * 10^{-11}\, t^3 \alpha_1 - 3.01805 * 10^{-14}\, t^4 \alpha_1 - 1.36096 * 10^{-10}\, t^3 \alpha_1{}^2 - 1.22707 * 10^{-12}\, t^4 \alpha_1{}^2 - t(-0.00165353\alpha_1 + 0.003215\alpha_2) - 1.31811 * 10^{-10}\, t^3 \alpha_2 + \cdots$

That

$u_1(t) = -\dfrac{\lambda_{14}(t)}{I_1} = -0.0115955(0.743687 - 0.00012251\, t + 5.03449 * 10^{-7} t^2 + 8.58655 * 10^{-10} t^3 - 7.81513 * 10^{-13} t^4),$

$u_2(t) = -\dfrac{\lambda_{24}(t)}{I_2} = -0.011755(0.361736 + 0.000233517\, t - 1.04679 * 10^{-6} t^2 - 8.36735 * 10^{-10} t^3 + 2.54377 * 10^{-13} t^4),$

$u_3(t) = -\dfrac{\lambda_{34}(t)}{I_3} = -0.00880359(0.120619 + 0.000066725\, t - 3.17293 * 10^{-7} t^2 - 1.53805 * 10^{-10} t^3 + 3.94016 * 10^{-13} t^4),$

with the final state condition $x_{14}(100) = 0, x_{24}(100) = 0,$ $x_{34}(100) = 0,$ we have gained:

$\alpha_1 = 0.7436866718467056,$

$\alpha_2 = 0.36173553498082317,$

$\alpha_3 = 0.12061935192581671.$

We consider $\varepsilon = 10^{-4}$. By applying the proposed method, the numerical results for J_i and stopping criterion are as given in Table III. The maximum absolute error of the proposed method, SHAM Chebyshev [14], SHAM Legendre [14] and HPM are as given in Table IV. It is noteworthy that the given method improves the maximum absolute error which indicates the efficiency of the method. Also, the obtained numerical solution for $x(t)$ and $u(t)$ in four iterative are depicted in Figures 2, 3 and 4.

Table III: Numerical results for different iteration, Example 2

i	1	2	3	4
J_i	0.00468052	0.00467797	0.00467903	0.00467886
$\left\|\dfrac{J_i - J_{i-1}}{J_i}\right\|$	-	5.451082 × 10^{-4}	2.265426× 10^{-4}	3.63336× 10^{-5}
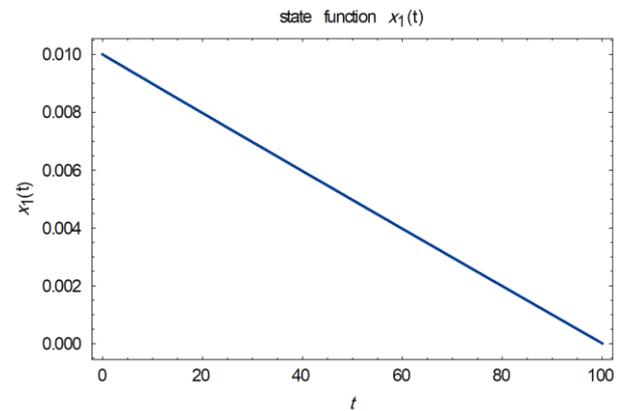
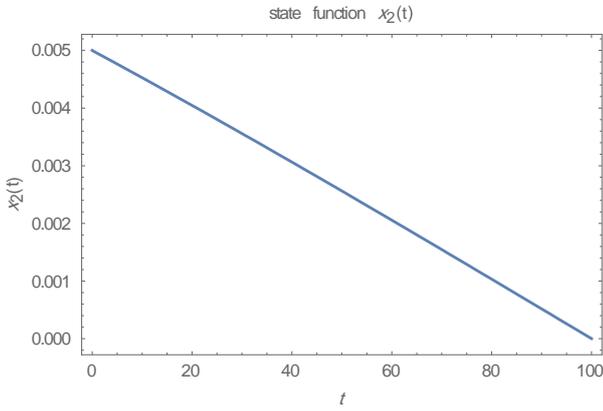**Fig. 2**(a). Suboptimal state $x_1(t)$,Example 2.

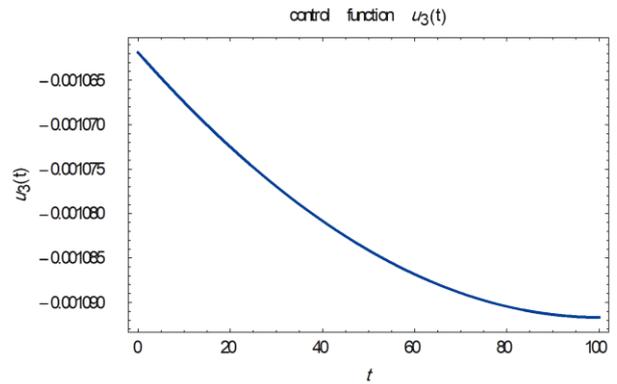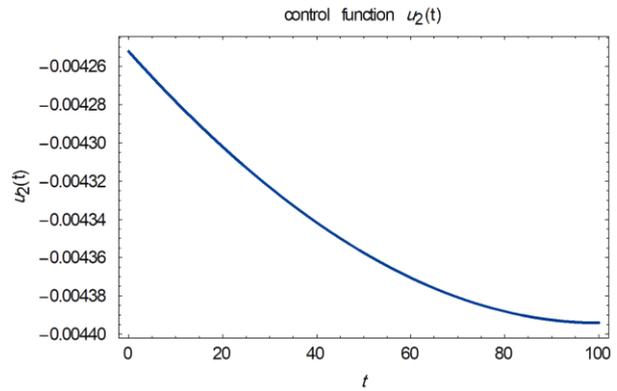**Fig. 2**: Suboptimal state $x_2(t)$, Example 2.

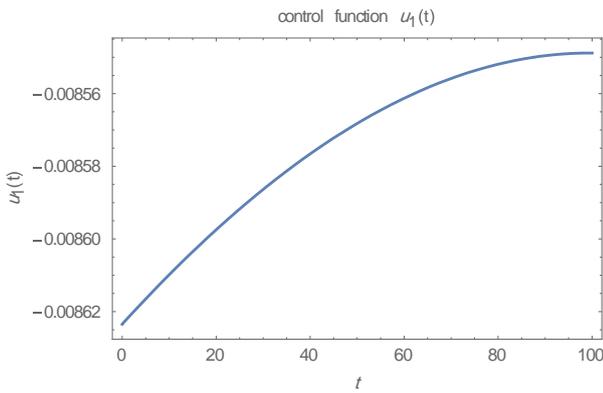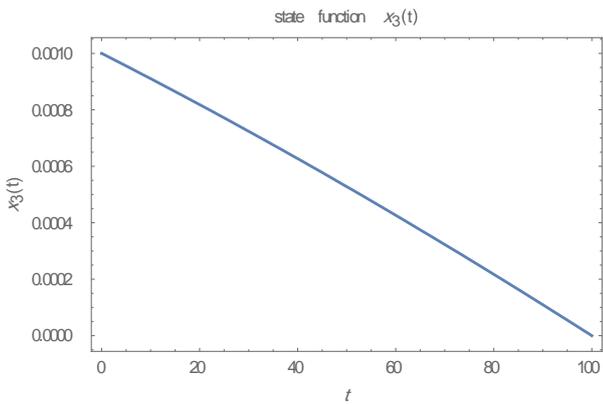**Fig. 4**: Suboptimal control $u_2(t)$ and $u_3(t)$, Example 2.

Fig. 3: Suboptimal control $x_3(t)$ and $u_1(t)$, Example2.

Table IV: Numerical results for the proposed method versus other methods, Example 2

METHOD	Objective value	Max state error	CPU time
PROPOSED METHOD	0.004678	2.40484×10^{-14}	0.046875
SHAM CHEBICHEV (M=6, N=50, H= -1.2)	0.004687	1.0586×10^{-9}	-
SHAM LEGENDER (M=6, N=50, H= -1.2)	0.004687	1.0589×10^{-9}	-
HPM (M=6)	0.004687795533	3.1420×10^{-8}	-

Example 3. Consider the non-linear system described by

$x_1' = x_2 + x_1 x_2$,

$x_2' = -x_1 + x_2 + x_2^2 + u,$

$x_1(0) = -0.8,$ $x_2(0) = 0,$

and the functional

$$minimize\ J = \frac{1}{2}\int_0^1 (x_1{}^2 + x_2{}^2 + u^2)dt.$$

The extreme conditions are

$\lambda_1' = -(x_1 + \lambda_1 x_2 - \lambda_2),$

$\lambda_2' = -(x_2 + \lambda_1(1 + x_1) + \lambda_2(1 + 2x_2)),$

$x_1' = x_2 + x_1 x_2,$

$x_2' = -x_1 + x_2 + x_2{}^2 - \lambda_2,$

that

$x_1(0) = -0.8, x_2(0) = 0,\ \lambda_1(1) = \lambda_2(1) = 0,$

and the optimal control is $u = -\lambda_2$. By using Mathematica software, two- term approximations for λ_1, λ_2, x_1 and x_2, were obtained as follows:

$x_{12}(t) = -0.8 + 0.08\ t^2 - 0.1\ t^2\alpha_1,$

$x_{22}(t) = 0.4\ t^2 + 0.1\ t^2\alpha_1 - t(-0.8 + \alpha_2),$

$\lambda_{12}(t) = \alpha_1 - 0.5t^2\ \alpha_1 - t(-0.8 - \alpha_2) - \dfrac{t^2\alpha_2}{2}$

$\qquad + \dfrac{1}{2}t^2\alpha_1\alpha_2,$

$\lambda_{22}(t) = -0.48t^2 + 0.1t^2\alpha_1 + \alpha_2 + 0.1t^2\alpha_2 + t^2\ \alpha_2{}^2 - t(0.2\alpha_1 + \alpha_2),$

That

$u(t) \simeq u_{22}(t) = -\lambda_{22} = -0.536036 + 0.257913t + 0.278123t^2.$

We consider $\varepsilon = 3 \times 10^{-2}$. Once the proposed method is applied, the numerical results for J_i and stopping criterion are as given in Table 6. The obtained numerical solution for $x(t)$ and $u(t)$ in three iterative are depicted in Fig 5 and 6.
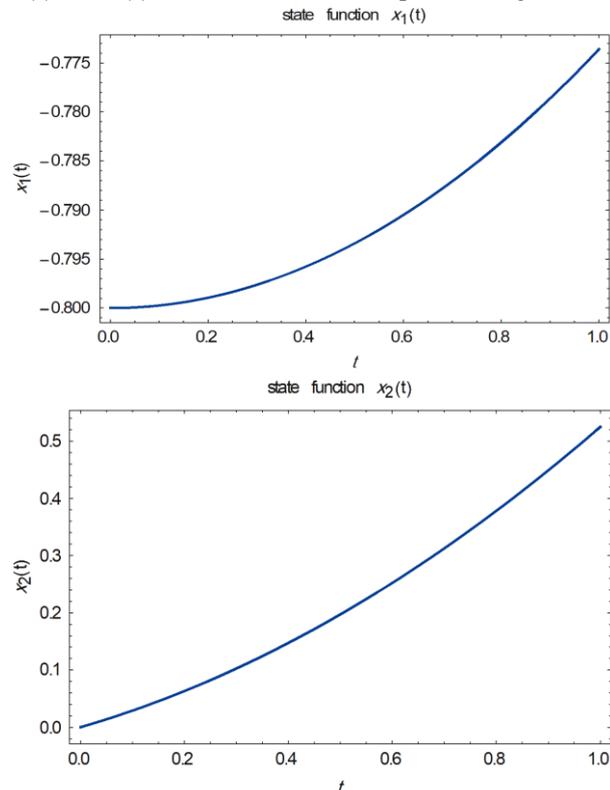
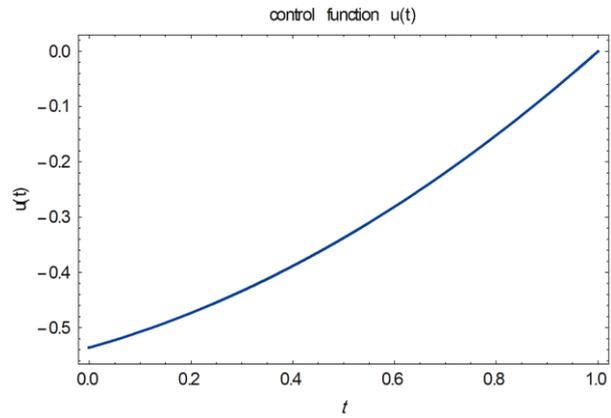**Fig 5**: Suboptimal state $x_1(t)$ and $x_2(t)$, Example 3.

Fig. 6: Suboptimal control $u(t)$, Example 3.

Table V: Calculation of α for $N = 2$, Example 3

α_1	α_2	j
-1.39061579	0.536036193	0.410274

Table VI: Numerical results for different iteration, Example 3

i	J_i	$\left|\frac{J_i - J_{i-1}}{J_i}\right|$
1	0.417241	-
2	0.410274	1.6981×10^{-2}
3	0.427145	3.9497×10^{-2}
4	0.431156	9.3028×10^{-3}

Table VII: Minimum of objective value for the proposed method, Example 3

Method	Objective value	CPU time (sec)
Proposed method (N=3)	0. 410274	0. 015625
VIM (N=3) [18]	0.4102	-

Conclusions

Due to its high computing demands, VIM cannot solve some non-linear optimal control problems. Hence, we proposed the modified variational iteration method to find a solution for this type of optimal control problems. The suggested method eliminates all additional calculations in VIM thus requires less time to do the computations. As stated, in Example1, the VIM cannot be counted more than two iterations due to additional calculations, but the proposed method is applicable in the high iteration. In addition, as seen in Table IV, the better max state error is obtained compared to other methods. As a future research direction, we can apply this method for solving optimal control problems ruled by partial differential equations and integral equations.

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