International Journal of Industrial Electronics, Control and Optimization © 2020 IECO
Vol. 3, No. 3, pp. 337-351, July (2020)

On the Coupling of Least Squares Method and Homotopy Perturbation Method for the Solution of Differential-Algebraic Equations and its Applications in Optimal Control Problems

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ABSTRACT

In this paper, two semi-analytical techniques are introduced to compute the solutions of differential-algebraic equations (DAEs), called the Least Squares Repetitive Homotopy Perturbation Method (LSRHPM) and the Least Squares Span Repetitive Homotopy Perturbation Method (LSSRHPM). The truncated series solution by the Homotopy Perturbation Method only is suitable for small-time intervals. Therefore, to extend it for long time intervals, we consider the Repetitive Homotopy Perturbation Method (RHPM). To improve the accuracy of the solutions obtained by RHPM and to reduce the residual errors, least squares methods and span set are combined with RHPM. The proposed methods are applied to solve nonlinear differential-algebraic equations and optimal control problems. The results of the proposed methods are compared using some illustrative examples. The results demonstrate the effectiveness and high accuracy of the new modifications. The effect of the parameters on the accuracy and performance of the methods is studied through some illustrative examples.

Article Info

Keywords:
Differential-Algebraic Equations, Least Squares Method, Span Set, Optimal Control, Semi-Analytical Homotopy Perturbation Method.

Article History:
Received 2019-10-02
Accepted 2020-02-16

I. INTRODUCTION

Most of the phenomena encountered in nature and technology are modeled using nonlinear differential equations, while their constraints and conditions are defined as algebraic equations. In this way, the system is modeled as differential-algebraic equations (DAEs). Therefore, it is very important to find a solution to these types of systems. In many cases, as an exact analytical solutions is not available, semi-analytical, approximate and numerical solutions of DAEs are computed.
In recent years, many numerical and semi-analytical methods have been proposed to solve DAEs. In [1], a numerical technique for solving DAEs is presented by employing the Laplace homotopy analysis method (LHAM). In [2], a numerical solution of linear and nonlinear DAEs is proposed. The method consists of expanding the required approximate solution as the elements of Chebyshev cardinal functions. The operational matrices are presented for the integration and product of these Chebyshev cardinal functions. By using these operational matrices together, a differential-algebraic equation can be transformed into a system of algebraic equations. In [3], a linear regular index 2 method is considered. Using a decoupling technique, initial condition and boundary conditions are properly formulated. Regular index 1 DAEs are obtained by the regularization method. In [4], the existence and uniqueness theory of the solution of the initial and boundary value problems for higher index DAEs are proposed. In [5], an

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index reduction technique is suggested for high index linear differential-algebraic equations. Also, their numerical solution is provided by the pseudospectral method. Some other index reduction techniques are given in [6]-[7]. Other numerical approaches including the Runge-Kutta method, specialized Runge-Kutta method and numerical solution of DAEs, using a multiquadric approximation, are given in [8]-[11]. The application of the series method is presented in [12] to find the solution of differential-algebraic equations system. In [13], the multirate implicit Euler method to semi-explicit DAEs of index-1 is extended, where the algebraic variables only provide slow dynamical changes that use three different strategies to realise the coupling between the slow and the fast subsystems. Then, assumptions are provided on the macro-step size and the micro-step size that a consistency order 1 can be proven for all three coupling strategies and respective differential variables. For semi-explicit DAEs, the usage of the coupled slowest-first approach seems favourable, since it is the only coupling strategy, where consistency order 1 is derived also for the algebraic variables. In [14], a new implementation of a semi-analytical iterative method is proposed by Temimi and Ansari, called TAM, for approximate solutions of differential algebraic equations that arises in many engineering and applied sciences applications. The solution is obtained with easily computed components without any restrictive assumptions to deal with the nonlinear terms. Error analysis of the approximate solution is studied using the absolute error and high convergence. In the article [15], an approach is proposed for constructing collocation-variation difference schemes with several collocation points for the numerical solution of the initial value problem in DAEs. These methods do not require the calculation of the projections onto the kernel of a matrix, the use of generalized inverses, or derivatives of input data. In paper [16], an approximate analytical solution is given for the bistable Allen–Cahn equation. The Allen-Cahn equation is a 2nd-order nonlinear parabolic partial differential equation representing some natural physical phenomenon.
The homotopy perturbation method and homotopy analysis method are used for finding the approximate solution. These methods do not need the use of any transformation, discretization, unrealistic restriction and assumption.

In [17], the behavior of systems of complex differential equations is investigated when some condition to the quality of the solutions is added. It extends the Gackstatter and Laine's results concerning complex differential equations to the systems of algebraic differential equations.

Another class of methods to solve DAEs is the class of the iterative methods, which have recently interested researchers. The homotopy perturbation method (HPM) [18], the Adomian decomposition method(ADM) [19]-[20], the variational iteration method (VIM) [21]-[23] and the homotopy analysis method (HAM) [24], as some of the iterative methods, have been used to solve the linear and nonlinear DAEs. Recently, the homotopy perturbation method is applied to the continuous-time model predictive control (MPC) problem [25]. Continuous-time MPC ends to an optimal control problem, where a system of differential-algebraic equations with boundary conditions must be solved. This problem is solved using HPM [25]. In [26], the nonplanar hydrodynamic equations of the present plasma model are reduced to the damped nonplanar Korteweg-de Vries (dnKdV) equation using the reductive perturbation method (RPM). This equation is used to investigate the characteristics and dynamics of dissipative nonplanar solitons in a collisional electronegative dusty plasma. It is known that the dnKdV equation does not admit an analytical solution due to the linear term (the ion-neutral collision term and the geometric term). Therefore, the homotopy perturbation method has been used to solve the equation.

Some authors have investigated the homotopy perturbation from a mathematical point of view. Their main objective was theoretically analyzing the method and to shown eventually that under certain circumstances the homotopy perturbation method converges to the exact desired solution, without a priori knowledge of the exact solution. Another objective has been to address the error estimate of the approximate solution. In [27]-[28], the convergence of the homotopy perturbation method is proved.

As is expressed in Section 4 and is shown in the examples of Section 6, the HPM provides the approximate solutions of DAEs only for short time intervals. So, to extend it to a longer time interval and to improve the accuracy of the approximate solutions, this paper proposes three new modifications of HPM, named repetitive homotopy perturbation method (RHPM), least squares repetitive homotopy perturbation method (LSRHPM) and least-squares span repetitive homotopy perturbation method (LSSRHPM). The presented examples show that with the same number of terms as in the original homotopy series, the accuracy of the approximate solutions of the DAEs was significantly improved using the proposed methods.

The structure of this paper is as follows. Section 2 introduces the general formulation of DAEs. Section 3 briefly explains the HPM. In Section 4, RHPM and LSRHPM methods are proposed. The LSSRHPM is presented in Section 5. Section 6. demonstrates the efficiency and effectiveness of proposed methods through some numerical examples. Also, in order to illustrate the application of the proposed methods in engineering, the optimal control problem is solved using these methods. The conclusion is given in the last section.

II. PROBLEM STATEMENT

A system of DAEs is one that consists of ordinary differential equations (ODEs) coupled with purely algebraic equations. In this paper, we consider the following form of DAEs with initial

values:

$$\begin{cases} \mathbf{F}\left(X(t),\dot{X}(t),Y(t),t\right)=0 \\ X(t_0)=X_0, Y(t_0)=Y_0 \end{cases}, t\in I=\left[t_0,t_f\right] \quad (1)$$

where $t\in\mathbb{R}$ is the independent variable, $X(t)=\left(x_1(t),...,x_n(t)\right)$, $Y(t)=\left(y_1(t),...,y_m(t)\right)$ and $\mathbf{F}: R^n\times R^n\times R^m\times R\to R^{n+m}$. The objective is to solve DAEs to obtain functions X(t) and Y(t) that satisfy the equations and the initial conditions of the system (1). Rewrite DAEs (1) in the form:

$$\begin{cases} \dot{X}(t)=K\left(X(t),\dot{X}(t),Y(t)\right)+F(t) \\ Y(t)=H\left(X(t),Y(t)\right)+G(t) \\ X(t_0)=X_0, Y(t_0)=Y_0 \end{cases}, t\in I=\left[t_0,t_f\right] \quad (2)$$

where $F(t)=\left(f_1(t),...,f_n(t)\right)$ and $G(t)=\left(g_1(t),...,g_m(t)\right)$ are analytical functions, $K: R^n\times R^n\times R^m\to R^n$ and $H: R^n\times R^m\to R^m$.

III. Homotopy Perturbation Method

In this section, we briefly review the HPM. The HPM was first proposed by He [29-30]. This technique is a coupling of the traditional perturbation method and the homotopy in topology; it divides a complex problem continuously into some simple iterative problem that can be easily solved.To illustrate the basic concepts of the HPM, consider the following nonlinear differential equation:

$$\begin{cases} L(u(t))+N(u(t))-f(t)=0 \\ B(u)=0 \end{cases} \quad t\in I=\left[t_0,t_f\right] \quad (3)$$

here L is a linear operator, N is a nonlinear operator, f(t) is a known function, u(t) is an unknown function and B denotes the boundary conditions.

The first step in the homotopy technique is to construct a homotopy function $v(t,p): I\times[0,1]\to\mathbb{R}$ which satisfies:

$$L(v)-L\left(u_0(t)\right)+pL\left(u_0(t)\right)+p\left[N(v)-f(t)\right]=0 \quad (4)$$

where $p\in[0,1]$ is an embedding parameter, v(t,p) is an unknown function, $u_0(t)$ is an initial approximation of the solution of Eq. (3) which satisfies the boundary conditions. Obviously, from Eq. (4), when $p=0$, $v(t,0)=u_0(t)$ and when $p=1, v(t,0)=u_0(t)$. Thus, as p increases from 0 to 1, the solution $v(t,p)$ varies from $u_0(t)$ to the solution $u(t)$.

In the HPM, the approximate solutions of DAEs are obtained in series form. Assume that the solution of Eq. (3) is written as a power series in p:

$$v(t,p)=u_0(t)+\sum_{i\geq 1}p^i u_i(t) \quad (5)$$

Substituting (5) in (4), collecting the same powers of p and equating each coefficient of the powers of p with zero, we obtain:

$$\begin{aligned} &L\left(u_i(t)\right)=-N_{i-1}\left(u_0(t),u_1(t),\ldots,u_{i-1}(t)\right),\ i\geq 1, \\ &B\left(u_i(t)\right)=0 \end{aligned} \quad (6)$$

where $N_i, i\geq 0$ are the coefficients of p^i in the nonlinear operator N:

$$\begin{aligned} &N\left(u(t)\right)=N_0\left(u_0(t)\right)+pN_1\left(u_0(t),u_1(t)\right)+ \\ &p^2N_2\left(u_0(t),u_1(t),u_2(t)\right)+\ldots \end{aligned} \quad (7)$$

We remark that $u_i(t), i\geq 1$ is obtained from the linear Eq.(6), which are easily solved together with the boundary conditions. Therefore, according to the HPM, the s-term approximate solution of Eq. (3) can be expressed as:

$$u(t)\cong\hat{u}(t)=u_0(t)+\sum_{i=1}^{s-1}u_i(t), t\in I=\left[t_0,t_f\right] \quad (8)$$

IV. Least Squares Repetitive Homotopy Perturbation Method

In [31], a new method based on the HPM and the least squares method (LSM) is proposed to solve the nonlinear differential equations. In this section, we propose a new modification of this method to solve DAEs. In the first step, the RHPM algorithm is used to improve the solutions of DAEs by the HPM. Then, the LSM is applied to increase the accuracy of the solutions.

A. *Repetitive Homotopy Perturbation Method*

In most cases of solving a differential, algebraic or differential-algebraic equations by the HPM, the first term of the homotopy series is considered as a constant function equal to the initial value given in the system. On the other hand, the nonlinear terms replaced by the Taylor series expansion with order v around the initial time and the initial value. Therefore, the HPM provides the optimal solution just only in the neighbourhood of the initial time.

In order to overcome this difficulty, we modify the HPM to the RHPM in the following.

Consider the Eq. (3) First, choose the time step $\Delta>0$, and divide the time interval $I=\left[t_0,t_f\right]$ into subintervals $I_k=\left[t_{k-1},t_k\right], k=1,\ldots,z$, where $t_0<t_1<\ldots<t_z=t_f$, $t_k-t_{k-1}=\Delta$.

Then, we apply the HPM to each subinterval. In order to carry out the iterations in every subinterval $I_k=\left[t_{k-1},t_k\right]$, it is

required to select some initial values for the function $u_0(t)$ in that subinterval. But, in general, we do not have this information except at the initial point t_0. A simple way for obtaining initial value at each subinterval is to use the approximation of $u(t)$ at the last iteration of the preceding subinterval. In this way, the continuous approximate solution on the interval $\left[t_0, t_f\right]$ can be denoted as follows:

$$u(t) \cong \sum_{k=1}^{z} \chi_{I_k} \hat{u}^k(t) \tag{9}$$

Where $\hat{u}^k(t)$ is obtained by HPM for the Eq. (3) on $I_k = [t_{k-1}, t_k]$ and

$$\chi_{I_k} = \begin{cases} 1 & t \in I_k \\ 0 & t \notin I_k \end{cases}, k = 1, ..., z \tag{10}$$

B. Least-Squares Repetitive Homotopy Perturbation Method

Nowadays, the least-squares method is widely used to obtain the numerical values of the parameters to reduce the estimation error. To illustrate the LSRHPM, first rewrite DAEs (2) in an operator form:

$$L_{1i}\left(x_i(t)\right) + N_{1i}\left(X(t), \dot{X}(t), Y(t)\right) - f_i(t) = 0, i = 1, \ldots, n \tag{11}$$

$$L_{2j}\left(y_j(t)\right) + N_{2j}\left(X(t), Y(t)\right) - g_j(t) = 0, j = 1, .., m \tag{12}$$

and define

$$L_1 = \left(L_{11}, \ldots, L_{1n}\right), \quad L_2 = \left(L_{21}, \ldots, L_{2m}\right) \tag{13}$$

$$N_1 = \left(N_{11}, \ldots, N_{1n}\right), \quad N_2 = \left(N_{21}, \ldots, N_{2m}\right) \tag{14}$$

where the differential and algebraic operators L_1 and L_2 are the linear operators and N_1 and N_2 are the nonlinear operators. Then, the s-term homotopy solutions of (11) and (12) on $I_k = [t_{k-1}, t_k]$, $k = 1, ..., z$ denote as following, which can be obtained using the HPM:

$$x_i^k(t) \cong \hat{x}_i^k(t) = x_{0i}^k(t) + \sum_{r=1}^{s-1} x_{ri}^k(t), i = 1, \ldots, n \tag{15}$$

$$y_j^k(t) \cong \hat{y}_j^k(t) = y_{0j}^k(t) + \sum_{r=1}^{s-1} y_{rj}^k(t), j = 1, \ldots, m \tag{16}$$

In the proposed LSRHPM, in each subinterval I_k, $k = 1, ..., z$ the core terms x_{ri}^k and y_{rj}^k, J_{LS}^k are maintained, however, their coefficients are tuned to improve the approximate solutions of DAEs. So, the homotopy series (15) and (16) are modified as follows:

$$\tilde{x}_i^k(t) = \alpha_{0i} x_{0i}^k + \sum_{r=1}^{s-1} \alpha_{ri} x_{ri}^k(t), \ i = 1, \ldots, n \tag{17}$$

$$\tilde{y}_j^k(t) = \beta_{0j} y_{0j}^k + \sum_{r=1}^{s-1} \beta_{rj} y_{rj}^k(t), \ j = 1, \ldots, m \tag{18}$$

where the coefficients are unknown and constant. Let's define for each $k = 1, ..., z$:

$$\tilde{X}^k = \left(\tilde{x}_1^k(t), \ldots, \tilde{x}_n^k(t)\right) \tag{19}$$

$$\tilde{Y}^k = \left(\tilde{y}_1^k(t), \ldots, \tilde{y}_m^k(t)\right) \tag{20}$$

$$\begin{aligned} \alpha_i^k &= \left(\alpha_{0i}^k, \ldots, \alpha_{(s-1)i}^k\right), \quad i = 1, \ldots, n, \\ \beta_j^k &= \left(\beta_{0j}^k, \ldots, \beta_{(s-1)j}^k\right), \ j = 1, \ldots, m \end{aligned} \tag{21}$$

$$\alpha^k = \left(\alpha_1^k, \ldots, \alpha_n^k\right), \ \beta^k = \left(\beta_1^k, \ldots, \beta_m^k\right) \tag{22}$$

Then, let's define

$$\hat{X}^k = \left(\hat{x}_1^k(t), \ldots, \hat{x}_n^k(t)\right) \tag{23}$$

$$\hat{Y}^k = \left(\hat{y}_1^k(t), \ldots, \hat{y}_m^k(t)\right) \tag{24}$$

By replacing the approximate solutions $\hat{X}^k$ and $\hat{Y}^k$ obtained in Eq. (11) and (12), the residual errors RDH_i^k and RAH_j^k for $k = 1, ..., z$ is evaluated as:

$$\begin{aligned} &(RDH_i^k)\left(\hat{x}_i^k, t\right) = \\ &L_{1i}\left(\hat{x}_i^k(t)\right) + N_{1i}\left(\hat{X}^k, \dot{\hat{X}}^k, \hat{Y}^k\right) - f_i(t), \ i = 1, \ldots, n \end{aligned} \tag{25}$$

$$\begin{aligned} &(RAH_j^k)\left(\hat{y}_j^k, t\right) = \\ &L_{2j}\left(\hat{y}_j^k(t)\right) + N_{2j}\left(\hat{X}^k, \hat{Y}^k\right) - g_j(t), \ j = 1, \ldots, m \end{aligned} \tag{26}$$

Define, the estimation performance measure as the mean of the squared residual errors on the subinterval I_k, $k = 1, ..., z$:

$$JDH_i^k = \frac{1}{|I_k|} \int_{I_k} (RDH_i^k)^2(t)\, dt, i = 1, \ldots, n \tag{27}$$

$$JAH_j^k = \frac{1}{|I_k|} \int_{I_k} (RAH_j^k)^2(t)\, dt, j = 1, \ldots, m \tag{28}$$

Now, using the LSM, the unknown coefficients α^k are obtained in order to reduce the above residual errors. Let's define the residual errors $RDLS_i^k$ and $RALS_j^k$, and the performance measure $JDLS_i^k$ and $JALS_j^k$ for the approximate solutions $\tilde{X}^k$ and $\tilde{Y}^k$ by replacing them in Eq. (11)-(12) similar to (25)-(28).

Functions $\tilde{X}^k$ and $\tilde{Y}^k$ must also satisfy the initial conditions of the Eq. (11)-(12) on any subinterval I_k, $k = 1, ..., z$, i.e.:

$$\begin{cases} \tilde{X}^1(t_0)=X(t_0), \tilde{Y}^1(t_0)=Y(t_0) & on\ I_1=[t_0,t_1] \\ \tilde{X}^k(t_{k-1})=\tilde{X}^{k-1}(t_{k-1}), mu\tilde{Y}^k(t_{k-1})=\tilde{Y}^{k-1}(t_{k-1}) on I_k=[t_{k-1},t_k] \\ \quad k=2,...,z \end{cases} \tag{29}$$

Therefore, the following constrained multi-objective optimization problem must be solved:

$$\min_{\alpha^k,\beta^k} \begin{cases} JDLS_1^k \\ \vdots \\ JDLS_n^k \\ JALS_1^k \\ \vdots \\ JALS_m^k \end{cases} \tag{30}$$

$$s.t \begin{cases} \begin{cases} (CD)(\alpha^k,\beta^k) := \tilde{X}^k(t_{k-1}) - X(t_{k-1}) = 0 \\ (CA)(\alpha^k,\beta^k) := \tilde{Y}^k(t_{k-1}) - Y(t_{k-1}) = 0 \end{cases} & if\ k=1 \\ \begin{cases} (CD)(\alpha^k,\beta^k) := \tilde{X}^k(t_{k-1}) - \tilde{X}^{k-1}(t_{k-1}) = 0 \\ (CA)(\alpha^k,\beta^k) := \tilde{Y}^k(t_{k-1}) - \tilde{Y}^{k-1}(t_{k-1}) = 0 \end{cases} & if\ k \geq 2 \end{cases}$$

There are different methods to solve this multi-objective optimization problem [32]. In this paper, the weighted sum method is used. By defining some weighting vector w as:

$$w=(w_{D1},\ldots,w_{Dn},w_{A1},\ldots,w_{Am}) \tag{31}$$

the multi-objective problem (30) becomes a single-objective problem:

$$\min_{\alpha^k,\beta^k} J_{LS}^k = w_{D1}(JDLS_1^k)+\ldots+w_{Dn}(JDLS_n^k)+w_{A1}(JALS_1^k)+ \ldots+w_{Am}(JALS_m^k) \tag{32}$$

$$s.t \begin{cases} \begin{cases} (CD)(\alpha^k,\beta^k) := \tilde{X}^k(t_{k-1}) - X(t_{k-1}) = 0 \\ (CA)(\alpha^k,\beta^k) := \tilde{Y}^k(t_{k-1}) - Y(t_{k-1}) = 0 \end{cases} & if\ k=1 \\ \begin{cases} (CD)(\alpha^k,\beta^k) := \tilde{X}^k(t_{k-1}) - \tilde{X}^{k-1}(t_{k-1}) = 0 \\ (CA)(\alpha^k,\beta^k) := \tilde{Y}^k(t_{k-1}) - \tilde{Y}^{k-1}(t_{k-1}) = 0 \end{cases} & if\ k \geq 2 \end{cases}$$

This problem is solved with the interior point method (IPM). Suppose that α^{k*} and β^{k*} are the optimal solutions of the problem (32) obtained by IPM. The approximate solutions of DAEs (11) and (12) on subinterval $I_k, k=1,...,z$ by LSRHPM are as follows:

$$x_i^k(t) \cong \tilde{x}_i^k(t)\big|_{\alpha_i^{k*}}, i=1,\ldots,n \tag{33}$$

$$y_j^k(t) \cong \tilde{y}_j^k(t)\big|_{\beta_j^{k*}}, j=1,\ldots,m \tag{34}$$

Lemma 1 *The optimization problem (32) has a feasible solution.*

Proof Put $\alpha^k=\mathbf{1}\in R^{sn}$ and $\beta^k=\mathbf{1}\in R^{sm}$, $k=1,...,z$ in the approximate solutions (17) and (18). So, we have $\tilde{X}^k=\hat{X}^k$ and $\tilde{Y}^k=\hat{Y}^k$. On the other hand, $\hat{X}^k$ and $\hat{Y}^k$ must satisfy the initial conditions (29) of the Eq. (10)-(11). Therefor $\alpha^k=\mathbf{1}\in R^{sn}$ and $\beta^k=\mathbf{1}\in R^{sm}$ are feasible solutions of (32). □

Definition *For each of the weighting vector w the value $J_H^k, k=1,...,z$:*

$$J_H^k = w_{D1}(JDH_1^k)+\ldots+w_{Dn}(JDH_n^k)+ w_{A1}(JAH_1^k)+\ldots+w_{Am}(JAH_m^k) \tag{35}$$

is called the sum of the mean of the squared residual errors of the approximate solutions of DAEs (11) and (12) obtained by RHPM in the subinterval I_k.

Theorem 1 *Let α^{k*} and β^{k*} are the optimal solutions of the problem (32). Then, for every $w\in R^{n+m}$, the optimal value of the performance measure J_{LS}^k is less than or equal to J_H^k, $k=1,...,z$.*

Proof Since $\alpha^k=\mathbf{1}\in R^{sn}$ and $\beta^k=\mathbf{1}\in R^{sm}$, $k=1,...,z$ are the feasible solutions to the problem (31), the value of the objective function of the problem for them is equal to J_H^k. We use the IPM which is guaranteed to decrease the performance criteria (32), therefore the optimal value (32) is equal or less than J_H^k, in other words, $J_{LS}^{k\,*} \leq J_H^k$.

V. LEAST SQUARES SPAN REPETITIVE HOMOTOPY PERTURBATION METHOD

In this section, we propose a method called the least-squares span repetitive homotopy perturbation method (LSSRHPM). The objective of this method is to improve the approximate solutions of DAEs obtained by the LSRHPM. In this method, first, the span sets are defined for the approximate solutions of DAEs obtained by the RHPM. Then, using the LSM, a combination of the functions of the span set is determined in order to reduce the residual errors.

To illustrate the LSSRHPM, assume the approximate solutions (15) and (16) are obtained by RHPM on subinterval I_k, $k=1,...,z$. First, define the sets $(SD)_i^k=\{\varphi_{1i}^k,\ldots,\varphi_{r_i i}^k\}$, $i=1,\ldots,n$ and $(SA)_j^k=\{\phi_{1j}^k,\ldots,\phi_{r_j j}^k\}$, $j=1,\ldots,m$ containing the span of the functions $\hat{X}^k$ and $\hat{Y}^k$. These are sets of linearly independent functions in the vector space of the

continuous functions on the interval I_k, such that each of the functions of $\hat{X}^k$ and $\hat{Y}^k$ is a real linear combination with real coefficients of these functions. So, the modified approximate solutions of (15) and (16) are defined as follow:

$$\tilde{\tilde{x}}_i^k(t)=\sum_{l=1}^{r_i}\delta_{li}^k\varphi_{li}^k, i=1,\ldots,n \tag{36}$$

$$\tilde{\tilde{y}}_j^k(t)=\sum_{l=1}^{r_j}\rho_{lj}^k\phi_{lj}^k, j=1,\ldots,m \tag{37}$$

where the coefficients are unknown and constant. Let's define for each $k=1,...,z$:

$$\tilde{\tilde{X}}^k=\left(\tilde{\tilde{x}}_1^k(t),\ldots,\tilde{\tilde{x}}_n^k(t)\right) \tag{38}$$

$$\tilde{\tilde{Y}}^K=\left(\tilde{\tilde{y}}_1^k(t),\ldots,\tilde{\tilde{y}}_m^k(t)\right) \tag{39}$$

$$\begin{aligned}&\delta_i^k=\left(\delta_{1i}^k,\ldots,\delta_{r_i i}^k\right),\quad i=1,\ldots,n\\&\rho_j^k=\left(\rho_{1j}^k,\ldots,\rho_{r_j j}^k\right),\ j=1,\ldots,m\end{aligned} \tag{40}$$

$$\delta^k=\left(\delta_1^k,\ldots,\delta_n^k\right),\rho^k=\left(\rho_1^k,\ldots,\rho_m^k\right) \tag{41}$$

Now, using the LSM, the unknown coefficients vectors δ^k and ρ^k are obtained in order to reduce the residual errors. Let's define the residual errors $RDLSS_i^k$ and $RALSS_j^k$, and the performance measure $JDLSS_i^k$ and $JALSS_j^k$ of the approximate solutions $\tilde{\tilde{X}}^k$ and $\tilde{\tilde{Y}}^k$ similar to 11-14 for each $k=1,...,z$.

When determining the unknown coefficients δ^k and ρ^k, functions $\tilde{\tilde{X}}^k$ and $\tilde{\tilde{Y}}^k$ must satisfy the initial conditions of Eq. (11)-(12) on subinterval $I_k, k=1,...,z$ i.e.:

$$\begin{cases}\tilde{\tilde{X}}^1(t_0)=X(t_0),\quad \tilde{\tilde{Y}}^1(t_0)=Y(t_0) & on\ I_1=[t_0,t_1]\\ \tilde{\tilde{X}}^k(t_{k-1})=\tilde{\tilde{X}}^{k-1}(t_{k-1}),\ \tilde{\tilde{Y}}^k(t_{k-1})=\tilde{\tilde{Y}}^{k-1}(t_{k-1}) & on I_k=[t_{k-1},t_k]\\ \quad k=2,...,z\end{cases} \tag{42}$$

Therefore, a multi-objective optimization problem similar to (30) must be solved. Using a weighting vector, the problem can be converted to the following single-objective problem:

$$\min_{\delta^k,\rho^k} J_{LSS}^k=w_{D1}(JDLSS_1^k)+\ldots+w_{Dn}(JDLSS_n^k)+ w_{A1}(JALSS_1^k)+\ldots+w_{Am}(JALSS_m^k)$$

$$s.t\begin{cases}\begin{cases}(CD)(\delta^k,\rho^k):=\tilde{\tilde{X}}^k(t_{k-1})-X(t_{k-1})=0\\(CA)(\delta^k,\rho^k):=\tilde{\tilde{Y}}^k(t_{k-1})-Y(t_{k-1})=0\end{cases} & if\ k=1\\ \begin{cases}(CD)(\delta^k,\rho^k):=\tilde{\tilde{X}}^k(t_{k-1})-\tilde{\tilde{X}}^{k-1}(t_{k-1})=0\\(CA)(\delta^k,\rho^k):=\tilde{\tilde{Y}}^k(t_{k-1})-\tilde{\tilde{Y}}^{k-1}(t_{k-1})=0\end{cases} & if\ k\geq 2\end{cases} \tag{43}$$

Suppose that δ^{k*} and ρ^{k*} are the optimal solutions of the problem (43) obtained by the IPM. Then, denote the approximate solutions of DAEs (11) and (12) on any subinterval I_k, $k=1,..,z$ by LSSRHPM as following:

$$x_i^k(t)\cong\tilde{\tilde{x}}_i^k(t)\big|_{\delta_i^{k*}}, i=1,\ldots,n \tag{44}$$

$$y_j^k(t)\cong\tilde{\tilde{y}}_j^k(t)\big|_{\rho_j^{k*}}, j=1,\ldots,m \tag{45}$$

Lemma 2 *The optimization problem (43) has a feasible solution.*

Proof Let $\hat{X}^k$ and $\hat{Y}^k$ are the approximate solutions of DAEs (11) and (12) on subinterval I_k, $k=1,..,z$ by RHPM, and $(SD)_i^k, i=1,..,n$ and $(SA)_j^k, j=1,..,m$ are their span set. It is clear that each of functions of $\hat{X}^k$ and $\hat{Y}^k$ is a linear combination of the functions $(SD)_i^k$ and $(SA)_j^k$. In other words, there exist $\delta_i^k=\left(\delta_{1i}^k,\ldots,\delta_{r_i i}^k\right), i=1,\ldots,n$ and $\rho_j^k=\left(\rho_{1j}^k,\ldots,\rho_{r_j j}^k\right), j=1,\ldots,m$ such that:

$$\hat{x}_i^k(t)=\sum_{l=1}^{r_i}\delta_{li}^k\varphi_{li}^k,\varphi_{li}^k\in(SD)_i^k, i=1,\ldots,n \tag{46}$$

$$\hat{y}_j^k(t)=\sum_{l=1}^{r_j}\rho_{lj}^k\phi_{lj}^k,\phi_{lj}^k\in(SA)_j^k, j=1,\ldots,m \tag{47}$$

Since $\hat{X}^k$ and $\hat{Y}^k$ satisfy the initial conditions (42) of Eq. (11)-(12). then $\delta^k=\left(\delta_1^k,...,\delta_n^k\right)$ and $\rho^k=\left(\rho_1^k,...,\rho_m^k\right)$ are feasible solutions to the problem (43).

Theorem 2 *Let δ^{k*} and ρ^{k*} are the optimal solutions of the problem (42). Then, for every $w\in R^{n+m}\geq 0$, $J_{LSS}^{k\ *}\leq J_{LS}^{k\ *}\leq J_H^k, k=1,...,z$.*

Proof To determine the approximate solutions $\tilde{\tilde{X}}^k$ and $\tilde{\tilde{Y}}^k$, $k=1,...,z$, first, $\hat{X}^k$, $\hat{Y}^k$, $(SD)_i^k$ and $(SA)_j^k$ are obtained. According to Lemma 1, it is clear that $J_{LSS}^{k\ *}\leq J_H^k$. Then, by modifying the coefficients of the homotopy series by LSRHPM, the approximate solutions $\tilde{X}^k$

and $\tilde{Y}^k$ are determine in order to satisfy the initial condition (42).

According to Theorem 1, we have $J_{LS}^{k\,*} \le J_H^k$. But, $\tilde{x}_i^k(t)\big|_{\alpha_i^{k*}} = \alpha_i^{k*}\hat{x}_i^k, i=1,\ldots,n$ and $\tilde{y}_j^k(t)\big|_{\beta_j^{k*}} = \beta_j^{k*}\hat{y}_j^k$, $j=1,\ldots,m$. So, there exist $a_i^k = \left(a_{1i}^k,\ldots,a_{r_i i}^k\right), i=1,\ldots,n,$ and $b_j^k = \left(b_{1j}^k,\ldots,b_{r_j j}^k\right),\ j=1,\ldots,m$ such that $\tilde{x}_i^k(t)\big|_{\alpha_i^{k*}} = \sum_{l=1}^{r_i} a_{li}^k\varphi_{li}^k$, $i=1,\ldots,n$ and $\tilde{y}_j^k(t)\big|_{\beta_j^{k*}} = \sum_{l=1}^{r_j} b_{lj}^k\phi_{lj}^k$, $j=1,\ldots,m$. Therefore the vectors $a^k=(a_1^k,\ldots,a_n^k)$ and $b^k=(b_1^k,\ldots,b_m^k)$ are feasible solutions to the problem (43). Using the IPM guarantees a decrease in the performance criteria (43). Hence, $J_{LSS}^{k\,*} \le J_{LS}^{k\,*}$ and the theorem is proved. □

VI. EXAMPLES AND APPLICATION

In this section, the ability and efficiency of the proposed methods are demonstrated to solve DAEs and their applications are investigated in optimal control problems (OCPs), through some numerical examples.

Example 1 Consider the following nonlinear initial value DAEs [18]:

$$\begin{cases} y' = y - zx + g_1(t) \\ z' = tx + y^2 + g_2(t) \\ 0 = y - x + g_3(t) \\ x(0) = y(0) = z(0) \end{cases}, \quad 0 \le t \le 1 \qquad (48)$$

where $g_1(t)=\sin(t)+t\cos(t)$, $g_3(t)=t\left(\cos(t)-\sin(t)\right)$ and $g_2(t)=sec^2(t)-t^2\left(\cos(t)+sin^2(t)\right)$. The exact solutions to this problem are $x(t)=t\cos(t)$, $y(t)=t\sin(t)$ and $z(t)=tan(t)$. In [18], the approximate solutions of the problem (48) are computed using the HPM, with $s=8$ and Taylor series expansion with $\nu=9$, as:

$$x(t)\cong\hat{x}(t)= t-\frac{1}{2}t^3+\frac{1}{24}t^5-\frac{1}{15}t^6-\frac{5}{144}t^7+\frac{323}{20160}t^8+\ldots \qquad (49)$$

$$y(t)\cong\hat{y}(t)= t^2-\frac{1}{6}+t^4\frac{1}{120}t^6-\frac{2}{105t}t^7-\frac{1159}{20160}t^8-\frac{443}{25920}t^9+\ldots \qquad (50)$$

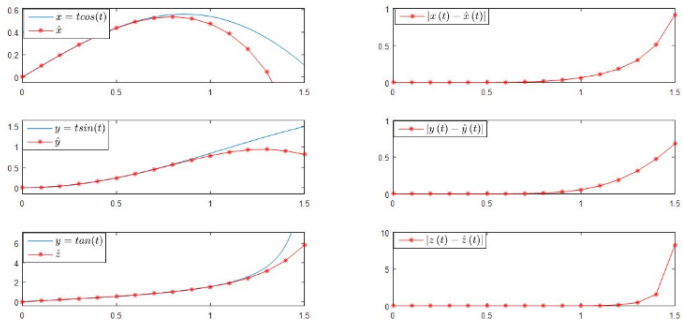


Fig. 1: Left: Exact and approximate solutions of example1,with $s=8$ and $\nu=9$ by HPM. Right: Absolute errors.

$$z(t)\cong\hat{z}(t)= t+\frac{1}{3}t^3+\frac{2}{15}t^5+\frac{2}{45}t^7-\frac{31}{720}t^8+\frac{139}{3240}t^9\ldots \qquad (51)$$

We want to investigate and compare the approximate solutions of problem (48) by the HPM and the proposed methods on an arbitrary long time interval, for instance $[0,1.5]$.

Fig.1 shows the exact solutions, the approximate solutions 49-51, and the absolute errors $\left|x(t)-\hat{x}(t)\right|$, $\left|y(t)-\hat{y}(t)\right|$ and $\left|z(t)-\hat{z}(t)\right|$. As can be seen in Fig. 1, the HPM provides suitable approximate solutions only for small time intervals $[0,1]$. Over that interval, the approximate solutions are far from the exact solutions.

To apply the proposed methods, by choosing the time step $\Delta=0.1$, Let's divide the interval $[0,1.5]$ into 15 subintervals, $\mathrm{I}_k=[\mathrm{t}_{k-1},\mathrm{t}_k], k=1,\ldots,15$. After that the HPM is applied on each subinterval with $s=2$ and the Taylor series expansion with $\nu=2$. Then, the approximate solutions by LSHPM and LSSHPM:

$$x(t)\cong\hat{x}(t)=x_0(t)+x_1(t) \qquad (52)$$

$$y(t)\cong\hat{y}(t)=y_0(t)+y_1(t) \qquad (53)$$

$$z(t)\cong\hat{z}(t)=z_0(t)+z_1(t) \qquad (54)$$

$$\tilde{g}_1(t)=\sum_{i=0}^{2}\frac{1}{i!}\left(\sin(t_0)+t_0\cos(t_0)\right)^{(i)}(t-t_0) \qquad (55)$$

$$\tilde{g}_2(t)=\sum_{i=0}^{2}\frac{1}{i!}\left(\sec^2(t_0)-t_0^2(\cos(t_0)+\sin^2(t_0))\right)^{(i)}(t-t_0) \qquad (56)$$

$$\tilde{g}_3(t)=\sum_{i=0}^{2}\frac{1}{i!}\left(t_0(\cos(t_0)+\sin(t_0))\right)^{(i)}(t-t_0) \qquad (57)$$

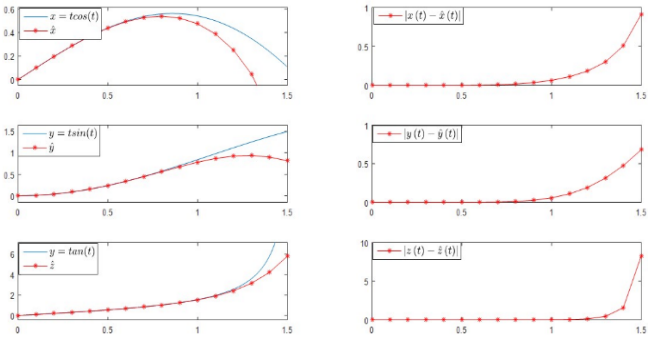


Fig. 2: Left: Exact and approximate solutions of example1, with $\Delta = 0.1$, $s = 2$ and $\nu = 2$ by RHPM, LSRHPM and LSSRHPM. Right: Comparison of absolute errors.

1.RHPM

The approximate solutions of example 1 on the interval $[0,1.5]$ by RHPM are:

$$\begin{cases} \hat{x}(t) = -t^2 + t \\ \hat{y}(t) = t^2 \\ \hat{z}(t) = t \end{cases} \quad on[0,0.1] \tag{58}$$

$$\begin{cases} \hat{x}(t) = -1.1396t^2 + 1.0136t + 0.0096 \\ \hat{y}(t) = -0.0665t^3 + 1.0100t^2 + 0.0003t - 0.0001 \\ \hat{z}(t) = 0.0037t^3 + 0.0443t^2 + 1.0002t - 0.0005 \end{cases} \quad on[0.1,0.2]$$

(59)

$$\begin{cases} \hat{x}(t) = -0.5846t^2 - 0.7962t + 2.6678 \\ \hat{y}(t) = -0.5324t^3 + 1.7162t^2 - 0.3409t + 0.1234 \\ \hat{z}(t) = 1176.1t^3 - 4739.5t^2 + 6390.7t - 2878.9 \end{cases} \quad on\,[1.4,1.5]$$

(60)

2. LSRHPM

The optimal coefficients and the approximate solutions by this method are:

$$\begin{cases} (\alpha^*, \beta^*) = (1.00001.06261.00001.00421.00001.0033) \\ \tilde{x}(t) = -1.0626t^2 + 1.0626t \\ \tilde{y}(t) = 1.0042t^2 \\ \tilde{z}(t) = 1.0033t \end{cases} \quad [0,0.1]$$

(61)

$$\begin{cases} (\alpha^*, \beta^*) = (0.94001.46031.00000.99851.00001.0090) \\ \tilde{x}(t) = -1.6641t^2 + 1.4801t - 0.0357 \\ \tilde{y}(t) = -0.0664t^3 + 1.0085t^2 - 0.0002t \\ \tilde{z}(t) = 0.0037t^3 + 0.0475t^2 + 1.0092t - 0.001 \end{cases} \quad [0.1,0.2]$$

(62)

$$\begin{cases} (\alpha^*, \beta^*) = (0.97320.49361.00000.98361.00001.0013) \\ \tilde{x}(t) = -0.2885t^2 - 0.3930t + 1.3588 \\ \tilde{y}(t) = -0.5236t^3 + 1.6880t^2 - 0.4240t + 0.1198 \\ \tilde{z}(t) = 1177.6t^3 - 4745.7t^2 + 6398.6t - 2882 \end{cases} \quad [1.4,1.5]$$

(63)

3. LSSRHPM

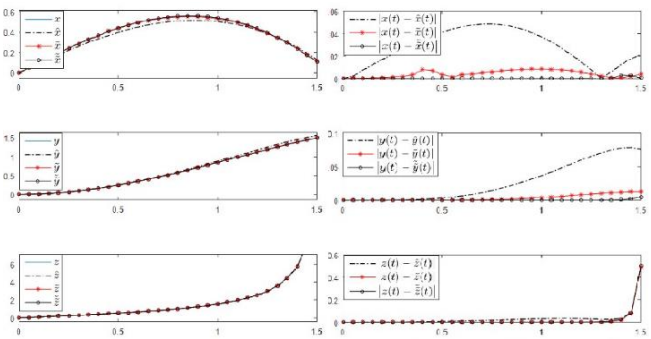


Fig. 3: Left: Exact and approximate solutions of example 1, with $\Delta = 0.05$, $s = 2$ and $\nu = 2$ by RHPM, LSRHPM and LSSRHPM. Right: Comparison of absolute errors.

The sets $\varphi_1 = \{\varphi_{11}, \ldots, \varphi_{1r_1}\}$, $\varphi_2 = \{\varphi_{21}, \ldots, \varphi_{2r_2}\}$ and $\varphi_3 = \{\varphi_{31}, \ldots, \varphi_{3r_3}\}$ are respectively the span sets of the approximate solutions $\hat{x}, \hat{y}$ and $\hat{z}$ obtained by HPM.

In each subinterval $I_k, k = 1, \ldots, z$ the span set, the optimal coefficients and the approximate solutions are:

$$\begin{cases} \varphi_1 = \{t, t^2\}, \varphi_2 = \{t, t^2\}, \varphi_3 = \{t\} \\ (\delta^*, \rho^*) = (0,1.0664, -0.9947, 0, 0, 0, 1.0041, 0, 0, 1.0033, 0, 0) \\ \tilde{\tilde{x}} = -0.9947t^2 + 1.0664t \\ \tilde{\tilde{y}} = 1.0041t^2 \\ \tilde{\tilde{z}} = 1.0033t \end{cases} \quad on[0,0.1]$$

(64)

$$\begin{cases} \varphi_1 = \{1, t, t^2\}, \varphi_2 = \{1, t, t^2, t^3\}, \varphi_3 = \{1, t, t^2, t^3\} \\ (\delta^*, \rho^*) = (-0.0234, 1.3071, -1.0585, 0, 0.0001, -0.0005, \\ \quad 1.0097, -0.0663, 0.001, 0.9794, 0.1369, 0.045) \\ \tilde{\tilde{x}} = -1.0585t^2 + 1.3071t - 0.0234 \\ \tilde{\tilde{y}} = -0.0663t^3 + 1.0097t^2 - 0.0005t + 0.0001 \\ \tilde{\tilde{z}} = 0.0459t^3 + 0.1369t^2 + 0.9794t + 0.0010 \end{cases} \quad on[0.1,0.2]$$

(65)

$$\begin{cases} \varphi_1 = \{1, t, t^2\}, \varphi_2 = \{1, t, t^2, t^3\}, \varphi_3 = \{1, t, t^2, t^3\} \\ (\delta^*, \rho^*) = (1.8288, -1.0944, -0.031136, 0, 0.050071, -0.39471, \\ \quad 1.7107, -0.53282, -2877.6, 6389.9, -4740.1, 1176.4) \\ \tilde{\tilde{x}} = -0.031136t^2 - 1.0944t + 1.8288 \\ \tilde{\tilde{y}} = -0.53282t^3 + 1.7107t^2 - 0.39471t + 0.050071 \\ \tilde{\tilde{z}} = 1176.4t^3 + -4740.1t^2 + 6389.9t - 2877.6 \end{cases} \quad on[1.4,1.5]$$

(66)

Fig. 2 demonstrates the exact and the approximate solutions of problem (48) on the interval $[0,1.5]$ by RHPM, LSRHPM and LSSRHPM and their absolute errors. Comparison of Fig. 2 demonstrates the exact and the approximate solutions of problem (48) on the interval $[0,1.5]$ by RHPM, LSRHPM and LSSRHPM and their absolute errors Comparison of

Figs. (1) and (2) reveals that the accuracy of the proposed methods significantly improves compare to the HPM.

In Table I, the absolute errors are given for the approximate solutions obtained by the three proposed methods in the time

Table I. Comparison of absolute errors of approximate solutions obtained by RHPM, LSRHPM and LSSRHPM of example 1 with Δ = 0.1, s = 2 and ν = 2.

t	RHPM			LSRHPM			LSSRHPM		
	$\lvert x(t)-x^{*}(t)\rvert$	$\lvert y(t)-y^{*}(t)\rvert$	$\lvert z(t)-z^{*}(t)\rvert$	$\lvert x(t)-x^{*}(t)\rvert$	$\lvert y(t)-y^{*}(t)\rvert$	$\lvert z(t)-z^{*}(t)\rvert$	$\lvert x(t)-\tilde{\tilde{x}}(t)\rvert$	$\lvert y(t)-\tilde{\tilde{y}}(t)\rvert$	$\lvert z(t)-\tilde{\tilde{z}}(t)\rvert$
0	0	0	0	0	0	0		0	0
0.1	0.0095	0.00001666	0.0003347	0.0024914	0.000016745	0.00000031074	0.002179	0.000040573	0.0000021215
0.2	0.02932	0.0001331	0.001341	0.0029857	0.000036267	0.0000074309	0.00067635	0.000059929	0.000003626
0.3	0.04846	0.0007793	0.003558	0.0061039	0.00006412	0.000031042	0.0012392	0.000075583	0.0000065124
0.4	0.06609	0.002456	0.007497	0.015961	0.00010299	0.000078557	0.00068258	0.00009437	0.0000255
0.5	0.08133	0.005737	0.01361	0.011858	0.0010026	0.00096058	0.0011201	0.00013739	0.00008488
0.6	0.09327	0.01125	0.02217	0.0085116	0.0011876	0.0010301	0.00028425	0.00016489	0.00016156
0.7	0.1009	0.01966	0.03316	0.0074428	0.0020504	0.0016504	0.000099699	0.00018807	0.00026121
0.8	0.1034	0.03158	0.04608	0.011111	0.0032196	0.0023293	0.0032803	0.00041911	0.0007628
0.9	0.09975	0.04749	0.05985	0.012564	0.0049794	0.003182	0.0025844	0.00057003	0.001292
1	0.0893	0.06757	0.07274	0.012423	0.0073266	0.0041289	0.010842	0.0014239	0.0025
1.1	0.07169	0.09152	0.08264	0.010205	0.010172	0.0054475	0.0076826	0.0034664	0.0050841
1.2	0.04709	0.1183	0.08835	0.0060203	0.013238	0.0090094	0.011051	0.0039628	0.009719
1.3	0.01645	0.1457	0.09643	0.00064339	0.016084	0.024982	0.005826	0.0045021	0.028281
1.4	0.01802	0.1693	0.1784	0.0050105	0.018195	0.12855	0.0022517	0.0088318	0.13629
1.5	0.05204	0.1804	1.794	0.013876	0.01828	1.7815	0.011132	0.012653	1.7827

points t_i, $i = 0,1,...,15$. In Table II, a comparison of the sum of the absolute errors is given for the approximate solutions of this example by HPM, RHPM, LSRHPM, and LSSRHPM. As a result, the following relation is observed in the time points t_i, $i = 0,1,...,15$:

$$\begin{aligned}&\left(\left|x(t_i)-\tilde{\tilde{x}}(t_i)\right|+\left|y(t_i)-\tilde{\tilde{y}}(t_i)\right|+\left|z(t_i)-\tilde{\tilde{z}}(t_i)\right|\right)<\\&\left(\left|x(t_i)-\tilde{x}(t_i)\right|+\left|y(t_i)-\tilde{y}(t_i)\right|+\left|z(t_i)-\tilde{z}(t_i)\right|\right)<\\&\left(\left|x(t_i)-\hat{x}(t_i)\right|+\left|y(t_i)-\hat{y}(t_i)\right|+\left|z(t_i)-\hat{z}(t_i)\right|\right)\end{aligned} \quad (67)$$

The results show LSSRHPM provides the most accurate solutions.

It is also important to notice that the degree of homotopy solution of HPM is 8 while it is 2 for all proposed methods. In Table 3, the sum of the mean squared residual errors is given in each subinterval by HPM, RHPM, LSRHPM, and LSSRHPM. It shows that in each subinterval I_k, $k = 1,...,z$, $J_{LSS}^{k\,*} \le J_{LS}^{k\,*} \le J_H^k$.

One of the important parameters in all proposed methods is the length of the subintervals, Δ. To study the effect of this parameter on the accuracy of the approximate solutions, the problem is also solved by the proposed methods with $\Delta = 0.05$. The results are given in Tables IV and V and Fig. 3. The comparison of Figs. 2 and 3 show that the accuracy of the approximate solutions can be increased by reducing the length of the subintervals.

Table II. Comparison of the sum of absolute errors of approximate solutions by HPM, RHPM, LSRHPM and LSSRHPM of example 1, with $\Delta = 0.1$, $s = 2$ and $\nu = 2$.

t	HPM Δ1.5 S=8, v=9	RHPM	LSHPM	LSSHPM
0	0	0	0	0
0.1	7.3665e-8	0.009852	0.0025084	0.0022217
0.2	5.2536e-6	0.0308	0.0030294	0.00073991
0.3	0.000066737	0.0528	0.006199	0.0013213
0.4	0.00041588	0.07604	0.016143	0.00080245
0.5	0.0017417	0.1007	0.013822	0.0013423
0.6	0.0056356	0.1267	0.010729	0.0006107
0.7	0.015181	0.1538	0.011144	0.00054898
0.8	0.035656	0.1811	0.01666	0.0044623
0.9	0.075491	0.2071	0.020726	0.0044464
1	0.14834	0.2296	0.023878	0.014766
1.1	0.27948	0.2458	0.025824	0.016233
1.2	0.52755	0.2537	0.028268	0.024733
1.3	1.0656	0.2585	0.04171	0.038609
1.4	2.5677	0.3658	0.15175	0.14738
1.5	9.8957	2.026	1.8137	1.8065

Table III. Comparison of the sum of mean of squared residual errors of $x(t)$, $y(t)$ and $z(t)$ on each subinterval by proposed methods example1, with $\Delta = 0.1$ and $w = (1,1,1)$.

I	J_{RHPM} $s = 2, \nu = 2$	J_{RLSHPM} $s = 2, \nu = 2$	$J_{RLSSHPM}$ $s = 2, \nu = 2$
1	0.00003721	0.00001012	0.0000096617
2	0.0003397	0.00003771	0.00000096989
3	0.0010662	0.00009287	0.00000077662
4	0.0024184	0.0002047	0.00000067472
5	0.0047601	0.001158	0.0000025289
6	0.0086401	0.0005546	0.00000071587
7	0.014793	0.001246	0.00000020777
8	0.024101	0.001725	0.000021673
9	0.037517	0.002427	0.0000095465
10	0.055981	0.003029	0.00010222
11	0.080503	0.003369	0.00026825
12	0.113	0.003249	0.000046301
13	0.16017	0.002579	0.00003691
14	0.25302	0.001506	0.00016277
15	0.60757	0.0005964	0.00019747
$J_{HPM} = 5.2355 with\ s = 8,\ \nu = 9, \Delta = 1.5$			

Table IV. Comparison of absolute errors of approximate solutions obtained by RHPM, LSRHPM and LSSRHPM of example 1 with Δ = 0.05, s = 2 and ν = 2.

t	RHPM			LSRHPM			LSSRHPM		
	$\lvert x(t)-\hat{x}(t)\rvert$	$\lvert y(t)-\hat{y}(t)\rvert$	$\lvert z(t)-\hat{z}(t)\rvert$	$\lvert x(t)-\tilde{x}(t)\rvert$	$\lvert y(t)-\tilde{y}(t)\rvert$	$\lvert z(t)-\tilde{z}(t)\rvert$	$\lvert x(t)-\tilde{\tilde{x}}(t)\rvert$	$\lvert y(t)-\tilde{\tilde{y}}(t)\rvert$	$\lvert z(t)-\tilde{\tilde{z}}(t)\rvert$
0	0	0	0	0	0	0	0	0	0
0.1	0.0074263	8.3303e-6	0.00015597	0.00049782	2.3174e-6	3.0208e-7	0.00032809	1.9305e-6	1.5383e-7
0.2	0.017223	0.00014719	0.00072498	0.001067	6.7455e-6	3.406e-6	0.00023326	4.1337e-6	3.0096e-8
0.3	0.026431	0.00064802	0.0019485	0.0021644	0.000020801	0.000017069	0.00022847	6.4239e-6	7.1669e-7
0.4	0.034611	0.0017751	0.0040577	0.0079488	0.00004886	0.000044098	0.00012723	9.0124e-6	2.8211e-6
0.5	0.041299	0.0038212	0.0072281	0.0037749	0.0002891	0.00025229	0.000068808	0.000011403	6.8999e-6
0.6	0.046019	0.0070929	0.011516	0.0036573	0.00055318	0.00043714	0.000033308	0.000013976	0.000014944
0.7	0.048309	0.011881	0.016787	0.0052587	0.0010006	0.00070509	0.000037134	0.00001744	0.000030589
0.8	0.047757	0.018413	0.02264	0.0068856	0.0017347	0.0010612	0.00010739	0.000022236	0.000060171
0.9	0.044068	0.026783	0.028351	0.0080876	0.00285	0.0014532	0.00017115	0.000029375	0.00011789
1	0.03714	0.036868	0.032865	0.0084933	0.0044096	0.0017648	0.00023533	0.00004092	0.00023937
1.1	0.027158	0.048225	0.034883	0.0077586	0.0064029	0.0018505	0.00030463	0.000061241	0.00052853
1.2	0.014706	0.059976	0.033203	0.0057151	0.0086909	0.0017557	0.00032736	0.00010622	0.0013599
1.3	0.00090539	0.070624	0.028168	0.0026055	0.010956	0.0029707	0.0000096997	0.00024734	0.0045578
1.4	0.012321	0.077627	0.033932	0.00069526	0.012715	0.020399	0.0031938	0.0011075	0.025253
1.5	0.021463	0.075789	0.48824	0.0039925	0.013395	0.49149	0.00060277	0.0046446	0.50024

Table V. Comparison of the sum of absolute errors of approximate solutions by HPM, RHPM, LSRHPM and LSSRHPM of $x(t)$, $y(t)$ and $z(t)$ of example 1, with $\Delta = 0.05$, $s = 2$ and $\nu = 2$.

t	HPM	RHPM	LSRHPM	LSSRHPM
0	0	0	0	0
0.1	7.3665e-8	0.0075906	0.00050044	0.00033017
0.2	5.2536e-6	0.018095	0.0010771	0.00023743
0.3	0.000066737	0.029028	0.0022022	0.00023561
0.4	0.00041588	0.040444	0.0080417	0.00013906
0.5	0.0017417	0.052348	0.0043163	0.000087111
0.6	0.0056356	0.064629	0.0046476	0.000062228
0.7	0.015181	0.076977	0.0069643	0.000085163
0.8	0.035656	0.08881	0.0096815	0.00018979
0.9	0.075491	0.099202	0.012391	0.00031841
1	0.14834	0.10687	0.014668	0.00051562
1.1	0.27948	0.11027	0.016012	0.00089441
1.2	0.52755	0.10788	0.016162	0.0017935
1.3	1.0656	0.099697	0.016532	0.0048149
1.4	2.5677	0.12388	0.033809	0.029555
1.5	9.8957	0.58549	0.50888	0.50549

Example 2 In the optimal control problems, the dynamic of the actual process is introduced as a system of differential equations, as well as algebraic equations which are the constraints and the boundary conditions. The objective of solving OCP is to find the control function and the corresponding state function which minimize some objective function. As a result, we have a differential optimization problem. One of the methods to solve this problem is the indirect method, which forms the necessary and sufficient optimality conditions by the calculus of variations and the Pontryagin's Minimum Principle (PMP). These conditions are differential-algebraic equations with boundary conditions.

Consider the following single input single output time-invariant linear quadratic problem [33]:

$$\mathbf{J} = \max_{u} \frac{1}{2}\int_0^1 x^2(t)+u^2(t)dt \qquad (68)$$

$$s.t\ \ \dot{x}(t) = -x(t)+u(t), x(0)=1$$

The corresponding Hamiltonian function will be

$$H\left(x,u,t\right) = \frac{1}{2}x^2(t)+\frac{1}{2}u^2(t)+\lambda(t)\left(-x(t)+u(t)\right) \qquad (69)$$

The optimal solutions should satisfy the following differential-algebraic equations with boundary conditions:

$$\begin{cases} \dot{x}(t) = -x(t)+u(t) \\ \dot{\lambda}(t) = \lambda(t)-x(t) \\ 0 = \lambda(t)+u(t) \\ x(0)=1, \lambda(1)=0 \end{cases} \qquad (70)$$

The system is DAEs with a two-point boundary value conditions. To solve this, let's first consider $\lambda(0)=\alpha$, which will be determined after solving (70).

The exact solutions are:

$$x(t) = \frac{\sqrt{2}\cosh\left(\sqrt{2}\left(t-1\right)\right)-\sinh\left(\sqrt{2}\left(t-1\right)\right)}{\sqrt{2}\cosh\left(\sqrt{2}\right)+\sinh\left(\sqrt{2}\right)}$$

and

$$\lambda(t) = -\frac{\sinh\left(\sqrt{2}\left(t-1\right)\right)}{\sqrt{2}\cosh\left(\sqrt{2}\right)+\sinh\left(\sqrt{2}\right)}$$

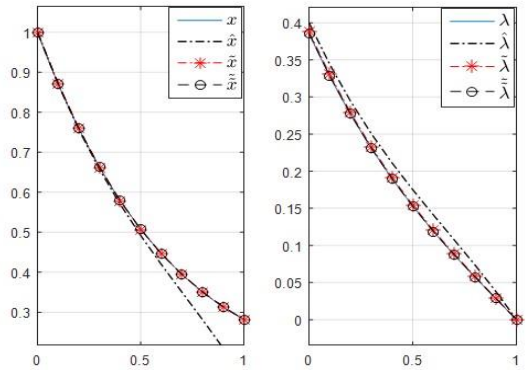


Fig. 4: Exact and approximate solutions by HPM, LSHPM and LSSHPM of example 2, with $s=4$.

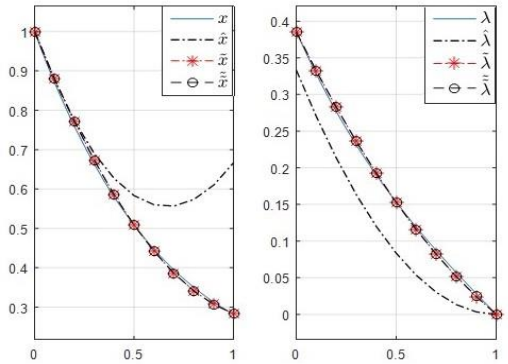


Fig.5: Exact and approximate solutions by HPM, LSHPM and LSSHPM of example2 with $s=3$.

Table VI. Comparison of absolute errors, the sum of absolute errors, objective functional value, the absolute error of **J** and sum of squared residual errors by HPM, LSHPM and LSSHPM with s = 4 of example 2.

t	HPM		LSHPM		LSSHPM	
	$\lvert x(t)-\hat{x}(t)\rvert$	$\lvert\lambda(t)-\hat{\lambda}(t)\rvert$	$\lvert x(t)-\tilde{x}(t)\rvert$	$\lvert\lambda(t)-\tilde{\lambda}(t)\rvert$	$\lvert x(t)-\tilde{\tilde{x}}(t)\rvert$	$\lvert\lambda(t)-\tilde{\tilde{\lambda}}(t)\rvert$
0	0	0.014181	0	3.3618e-7	0	4.4702e-6
0.1	0.0014391	0.01574	0.00086205	0.00030752	0.00086246	0.00028683
0.2	0.0031267	0.017526	0.00047876	0.00019169	0.00047916	0.00016637
0.3	0.0056274	0.019366	0.00029399	0.000055061	0.00029395	0.000076761
0.4	0.0098109	0.020973	0.00090510	0.00025179	0.00090573	0.00026456
0.5	0.0168130	0.021969	0.0010696	0.00031033	0.0010711	0.00031185
0.6	0.0280010	0.021900	0.00073509	0.00021888	0.00073781	0.00020978
0.7	0.0449480	0.020248	0.000053526	0.000027044	0.000057573	0.000010939
0.8	0.0694060	0.016451	0.00064386	0.00016785	0.00063836	0.00018438
0.9	0.1032800	0.009909	0.00086729	0.0002335	0.00086026	0.00024092
1	0.1486400	5.5511e-17	0.000013664	0.000014659	0.00002225	4.5193e-7
Sum absolute errors	0.6093563		0.007701532		0.00768858533	
J	0.184634921		0.192933614		0.19293301	
$\lvert\mathbf{J}^*-\mathbf{J}\rvert$	0.0082743771		0.0000243159		0.0000237119	
J(*sumresidual errors*)	0.07365079		0.00004884328		0.00004883993	

The optimal value of the objective functional (68) is $\mathbf{J}^*=0.1929092981$.

In this example, it is shown that we can improve the accuracy of the approximate solutions obtained by HPM by applying LSHPM and LSSHPM. The approximate solutions resulted from HPM, LSHPM, and LSSHPM with $s=4$ are shown in the following and in Fig. 4.

1- HPM:

$$\begin{cases}\hat{x}(t)=-0.4666667t^3+t^2-1.4t+1\\ \hat{\lambda}(t)=-0.2t^3+0.4t^2-0.6t+0.4\end{cases} \quad (71)$$

2- LSHPM:

$$\begin{cases}\tilde{x}(t)=-0.2299041t^3+0.8791291t^2-1.367269t+1\\ \tilde{\lambda}(t)=-0.1222263t^3+0.3441249t^2-0.6077029t+0.3858189\end{cases} \quad (72)$$

3- LSSHPM:

$$\begin{cases}\tilde{\tilde{x}}(t)=-0.2298962t^3+0.8791065t^2-1.367263t+1\\ \tilde{\tilde{\lambda}}(t)=-0.122243t^3+0.3441268t^2-0.6076968t+0.3858198\end{cases} \quad (73)$$

Fig. 4 and Table VI. show that $J^*_{LSS}\le J^*_{LS}\le J_H$. Also, the following results are obtained for the absolute error of the objective functional value of the problem (68), $\left|\mathbf{J}^*-\mathbf{J}_{LSS}\right|<\left|\mathbf{J}^*-\mathbf{J}_{LS}\right|<\left|\mathbf{J}^*-\mathbf{J}_H\right|$.

Table VII. Comparison of absolute errors, the sum of absolute errors, objective functional value, the absolute error of **J** and sum of squared residual errors by HPM, LSHPM and LSSHPM with $s = 3$ of example 2.

t	HPM		LSHPM		LSSHPM	
	$\lvert x(t) - \tilde{x}(t)\rvert$	$\lvert \lambda(t) - \hat{\lambda}(t)\rvert$	$\lvert x(t) - \tilde{x}(t)\rvert$	$\lvert \lambda(t) - \tilde{\lambda}(t)\rvert$	$\lvert x(t) - \tilde{\tilde{x}}(t)\rvert$	$\lvert \lambda(t) - \tilde{\tilde{\lambda}}(t)\rvert$
0	0	0.052485	0.000024902	0.00077318	0.000025418	0.00080167
0.1	0.0056943	0.05806	0.0092618	0.0040748	0.0092191	0.004017
0.2	0.01394	0.063541	0.011788	0.0055493	0.01171	0.0054719
0.3	0.026973	0.067901	0.0097895	0.0046756	0.0096828	0.0045886
0.4	0.046722	0.070227	0.0051965	0.0023681	0.0050677	0.0022814
0.5	0.074854	0.069697	0.00032668	0.00055177	0.00047072	0.00062819
0.6	0.1128	0.065567	0.0053482	0.0033388	0.0055007	0.0033951
0.7	0.16179	0.057152	0.0086404	0.0053092	0.0087948	0.0053354
0.8	0.22286	0.043816	0.0091552	0.0058269	0.0093047	0.0058132
0.9	0.29692	0.024958	0.006003	0.004291	0.0061409	0.0042275
1	0.3847	0	0.0015652	0.00012333	0.0014457	1.1271e-8
Sum absolute errors	1.92064373		0.103981678		0.103922679	
J	0.240740741		0.1949099		0.194843642	
|J* − J|	0.0478314429		0.0020006019		0.0019343439	
J(*residual errors*)	0.4444444		0.00356506672		0.0035648811	

The absolute errors of the obtained solutions by HPM, LSHPM, and LSSHPM using equal weight for the variables x, λ in the arbitrary points $t_0 = 0, t_1 = 0.1, ..., t_{10} = 1$ are calculated and the following relation is observed between the sum of the absolute errors by the three methods:

$$\sum_{i=0}^{10}\left(\left|x(t_i)-\tilde{\tilde{x}}(t_i)\right|+\left|\lambda(t_i)-\tilde{\tilde{\lambda}}(t_i)\right|\right) <$$

$$\sum_{i=0}^{10}\left(\left|x(t_i)-\tilde{x}(t_i)\right|+\left|\lambda(t_i)-\tilde{\lambda}(t_i)\right|\right) <$$

$$\sum_{i=0}^{10}\left(\left|x(t_i)-\hat{x}(t_i)\right|+\left|\lambda(t_i)-\hat{\lambda}(t_i)\right|\right)$$

To study the effect of the number of the terms in the homotopy series on the accuracy of the approximate solutions, the problem is also solved by the proposed methods with $s = 3$ and the results are provided in Table VII and Fig.5. The comparison between Figs.4 and 5 show that the accuracy of the approximate solutions can be increased by increasing the number of terms s.

Example 3 Consider the following multiple-input multiple-output continuous time-varying system:

$$\dot{x}(t) = f(x,u,t) = \begin{bmatrix} 0.25sint + 0.5 & 0 \\ 0 & -0.25sin2t - 0.5 \end{bmatrix} x(t) + \begin{bmatrix} 0.5 & 0 \\ 0 & 2 + sint \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} x(t) \tag{74}$$

with the initial states $x^T(0) = [0,1]$, its initial corresponding output $y^T(0) = [0,-1]$ and the reference signal $y_{ref}^T = [1,1]$. To track the reference signal by outputs, define the objective functional as follows:

$$J = \overbrace{\left[y(t_f) - y_{ref}(t_f)\right]^T \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \left[y(t_f) - y_{ref}(t_f)\right]}^{h} + \int_{t_0}^{t_f} \left[y(t) - y_{ref}(t)\right]^T \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \left[y(t) - y_{ref}(t)\right] + u^T \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} u dt \tag{75}$$

According to the optimal control theory, the optimal solutions should satisfy the following differential-algebraic equations with boundary conditions:

$$
\begin{cases}
\dot{x}_1(t) = (0.25sint + 0.5)x_1(t) + 0.5\left(\dfrac{-0.5}{2R_1}\lambda_1(t)\right) \\
\dot{x}_2(t) = (-0.25sin2t - 0.5)x_2(t) + (2+sint)\left(\dfrac{-(2+sint)}{2R_2}\lambda_2(t)\right) \\
\dot{\lambda}_1(t) = -2Q_1\left(x_1(t) - y_{1ref}(t)\right) + 2Q_2\left(-x_1(t) - x_2(t) - y_{2ref}(t)\right) \\
\qquad -0.25sint\lambda_1(t) - 0.5\lambda_1(t) \\
\dot{\lambda}_2(t) = 2Q_2\left(-x_1(t) - x_2(t) - y_{2ref}(t)\right) + 0.25sin2t\lambda_2(t) + 0.5\lambda_2(t) \\
u_1 = \dfrac{-0.5}{2R_1}\lambda_1(t) \\
u_2 = \dfrac{-(2+sint)}{2R_2}\lambda_2(t) \\
x(t_0) = x_0 \\
\left.\dfrac{\partial h}{\partial x_1}\right|_{t=t_f} = 2H_1\left(x_1(t_f) - y_{1ref}(t_f)\right) - \\
\qquad 2H_2\left(-x_1(t_f) - x_2(t_f) - y_{2ref}(t_f)\right) - \lambda_1(t_f) = 0 \\
\left.\dfrac{\partial h}{\partial x_2}\right|_{t=t_f} = -2H_2\left(-x_1(t_f) - x_2(t_f) - y_{2ref}(t_f)\right) - \lambda_2(t_f) = 0
\end{cases}
$$

$$
\begin{cases}
\dot{x}_1(t) = (0.25sint + 0.5)x_1(t) + 0.5\left(\dfrac{-0.5}{2R_1}\lambda_1(t)\right) \\
\dot{x}_2(t) = (-0.25sin2t - 0.5)x_2(t) + (2+sint)\left(\dfrac{-(2+sint)}{2R_2}\lambda_2(t)\right) \\
\dot{\lambda}_1(t) = -2Q_1\left(x_1(t) - y_{1ref}(t)\right) + 2Q_2\left(-x_1(t) - x_2(t) - y_{2ref}(t)\right) \\
\qquad -0.25sint\lambda_1(t) - 0.5\lambda_1(t) \\
\dot{\lambda}_2(t) = 2Q_2\left(-x_1(t) - x_2(t) - y_{2ref}(t)\right) + 0.25sin2t\lambda_2(t) + 0.5\lambda_2(t) \\
u_1 = \dfrac{-0.5}{2R_1}\lambda_1(t) \\
u_2 = \dfrac{-(2+sint)}{2R_2}\lambda_2(t) \\
x(t_0) = x_0 \\
\left.\dfrac{\partial h}{\partial x_1}\right|_{t=t_f} = 2H_1\left(x_1(t_f) - y_{1ref}(t_f)\right) - \\
\qquad 2H_2\left(-x_1(t_f) - x_2(t_f) - y_{2ref}(t_f)\right) - \lambda_1(t_f) = 0 \\
\left.\dfrac{\partial h}{\partial x_2}\right|_{t=t_f} = -2H_2\left(-x_1(t_f) - x_2(t_f) - y_{2ref}(t_f)\right) - \lambda_2(t_f) = 0
\end{cases}
\tag{76}
$$

Since we do not have the initial values of the costates $\lambda_1(t_0)$ and $\lambda_2(t_0)$, first, consider them as unknowns $\lambda_1(t_0) = \alpha_1$ and $\lambda_2(t_0) = \alpha_2$, which are finally calculated according to the boundary conditions. The results by RHPM and LSRHPM with $\Delta = 0.2$, $s = 2$ and $\nu = 2$ are shown in Fig. 6

From the results, it can be seen that the obtained outputs by LSRHPM track the reference signal better than outputs by RHPM. It should be noted that the system response oscillations are due to the time-varying nature of the system itself.

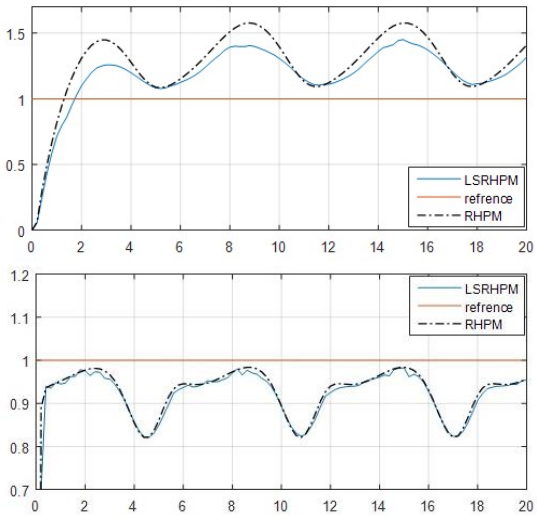


Fig. 6: Response of example 3 by RHPM and LSRHPM. up: output 1, down: output 2, with $\Delta = 0.2$, $s = 2$, $\nu = 2$.

VII. Conclusions

The motivation of this paper is to extend the semi-analytic homotopy perturbation method (HPM) to solve DAEs for large time intervals and to improve the accuracy of the approximate solutions. Therefore, the new modifications named the Repetitive Homotopy Perturbation Method, the Least Square Repetitive Homotopy Perturbation Method, and the Least Square Span Repetitive Homotopy Perturbation Method are proposed. These techniques were tested on the nonlinear DAEs and the optimal control problems. The results confirm better approximation, especially for long time intervals, while the order of the homotopy solution can be smaller. The effect of the selection of the order of homotopy series S and the subinterval length Δ on the accuracy of the approximate solutions was observed. This shows that the accuracy will be improved when the S increases or Δ decreases. Furthermore, compared to the HPM, only a small number of series terms are needed to obtain satisfactory solutions in the proposed methods.

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