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Numerical Method for Approximate Solutions of Fractional Differential Equations with Time-Delay

Fahimeh Akhavan Ghassabzadeh [1], Samaneh Soradi-Zeid [2,†]

[1] Department of Mathematics, Faculty of Sciences, University of Gonabad, Gonabad, Iran
[2] Faculty of Industry and Mining (Khash), University of Sistan and Baluchestan, Zahedan, Iran.

ABSTRACT

Due to the easy adaption of radial basis functions (RBFs), a direct RBF collocation method is considered to develop an approximate scheme to solve fractional delay differential equations (FDDEs). In spite of easy implementation of other high-order methods and finite difference schemes for solving a problem of fractional order derivatives, the challenge of these methods is their limited accuracy, locality, complexity and high cost of computing in discretization of the fractional terms, which suggest that global scheme such as RBFs that are more accurate way for discretizing fractional calculus and would allow us to remove the ill-conditioning of the system of discrete equations. So, the proposed approach, either employ any global RBFs for interpolating technique or uses arbitrary points for discretization, offers a very flexible framework for solving FDDEs. Applications to a variety of problems confirm that the proposed method is slightly more efficient than those introduced in other literatures and the convergence rate of our approach is high.

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I. INTRODUCTION

Using fractional order dynamics for diaphragms, bellows, and other devices that may have sprig-property yields a more accurate model for such industrial systems than the classical integer order models. In particular, relaxation processes deviating from the classical exponential behavior are often encountered in the dynamics of complex materials. In recent decay, the applications of fractional calculus in scientific fields such as power and environment sciences is improved and this has caused that the FDDEs have been review by many researchers due to their applications over the simulation and modeled in engineering, physics, hydrology, pure and applied mathematics [1, 2, 3, 4]. Many considerable studies on the theory, existence of solutions and numerical analysis of FDDEs have been presented in [5, 6, 7, 8, 9, 10]. Dehghan et al. used the concept of Legendre operational matrix to present the numerical solutions of FDDEs in [11]. Morgado [12], applied the Mittag-Leffler functions to approximate solutions for FDDEs. The Chebyshev wavelet method to compute an approximation to the solution of the FDDEs has been employed in [13]. Dehghan et al. [14] used the variational iteration and Adomian decomposition methods to obtain the approximate solutions for the delay logistic equation. Amar et al. in [15], concentrated on developing an efficient approximate solution of fractional delay quasi-linear control inclusion. Finite difference scheme is applied to solve the FDDEs in [16]. Wang [17] approximated the FDDEs by combining the general Adams-Bashforth-Moulton method with the linear interpolation method. In another paper, Wang et al. [18] introduced a numerical method for nonlinear functional order differential equations with constant time varying delay, based on Grunwald-Letnikov definition. Recently, Pandey et al. [19] applied the Bernstein's

†Corresponding Author: soradizeid@eng.usb.ac.ir
Tel: +98-54-33723560, University of Sistan and Baluchestan, Faculty of Industry and Mining (Khash), Zahedan, Iran.

operational matrix for the fractional derivative to solve two FDDEs of physical importance. Mohammadi in [20] used Chelyshkov-Wavelet to solve systems of FDDEs and a finite time stability results for FDDEs by virtue of delayed Mittag-Leffler type matrix present in [21]. The review article [22] contains an up-to-date bibliography on numerical methods for solving FDDEs.

The field of industry is a special application domain for fractional order systems as the demanding performance expectations with uncertainties make it a challenge to come up with models that are useful, or control modules that can alleviate the difficulties in various forms. The interest of the industry to fractional order systems lie in the fact that complicated modules can be simplified significantly and practical applications can be diverse. The investigation of industrial mathematics problems sometimes leads to the development of new methods of solution of differential equations. This special issue has covered both the theoretical and applied aspects of industrial mathematics. Papers contain the development of new mathematical models or well-known models applied to new physical situations as well as the development of new mathematical techniques [23, 24, 25, 26, 27]. This is bilaterally meaningful collaboration between mathematics and industry. Some literatures [28, 29, 30] consider the fractional order systems and control methods (such as PID controller, adaptive control and sliding mode control) within the context of industrial automation. The expectations of the industrial applications are demanding and often times the conventional solutions to problems are so complicated that the manufacturing of the goods based on standard approaches is not feasible. To overcome this challenge, we present a practical direct numerical method to solve these equations.

Radial basis techniques are one of the relatively new techniques used for finding solutions of fractional differential equations (FDEs). Due to better accuracy, stability, efficiency, memory requirement, and simplicity of implementation of RBFs over other methods, a number of researchers are being attracted toward these techniques. RBF meshless method used by Liu et al. in [31] for solving time fractional advection-diffusion equation with distributed order and by Dehghan et al. in [32] for solving the nonlinear time fractional Sine-Gordon and Klein-Gordon equations. Ahmadi et. al. applied RBFs for solving stochastic fractional differential equations in [33]. In addition, the local RBF method has been investigated in [34] to solve the variable-order time fractional diffusion equation. Also, RBFs are actively used for solving KdV equation in [35, 36]. A local RBFs collocation method based on the MQ RBFs for solving nonlinear coupled Burgers equations is presented in [37]. Kumar and Yadav in [38] provide RBF neural network techniques for solving differential equations of various kinds. Some other work related to this field may be found in [39, 40, 41]. Control systems can include both the fractional-order dynamic system to be controlled and the fractional-order controller. However, in the current paper, we focus on providing a numerical scheme based on RBFs to solve the following FDDE:

$$D^{\alpha}x(t) = a(t)x(t) + b(t)x(t-\tau) + f(t),$$
$$t\epsilon(0,T], \quad \tau > 0, \;\; n-1 < \alpha \le n$$
$$x(t) = \varphi(t), \quad -\tau \le t \le 0. \qquad (1)$$

where a, b and f are continuous functions on $[0,T]$, the initial function φ is a continuous function on $[-\tau,0]$ and D^{α} denotes the Caputo derivative of order α that was defined as follow:

Definition 1. The Caputo fractional derivative of order α is defined by

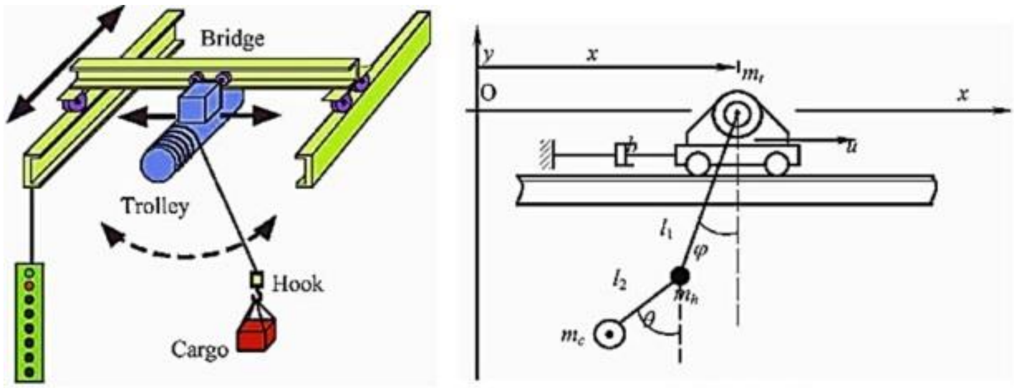
$$D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}(t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau,$$
$$n-1 < \alpha \le n, \qquad t > 0, \qquad (2)$$

where $\Gamma(.)$ is a Gamma function and D^{α} satisfies the following properties:

$$D^{\alpha}K = 0 \quad (K\ is\ a\ constant),$$
$$D^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha},\ \beta > \alpha-1 \qquad \big(when\ t_0 = 0\ in\ (2)\big)$$
$$D^{\alpha}\big(\lambda f(t) + \mu g(t)\big) = \lambda D^{\alpha}f(t) + \mu D^{\alpha}g(t). \qquad (3)$$

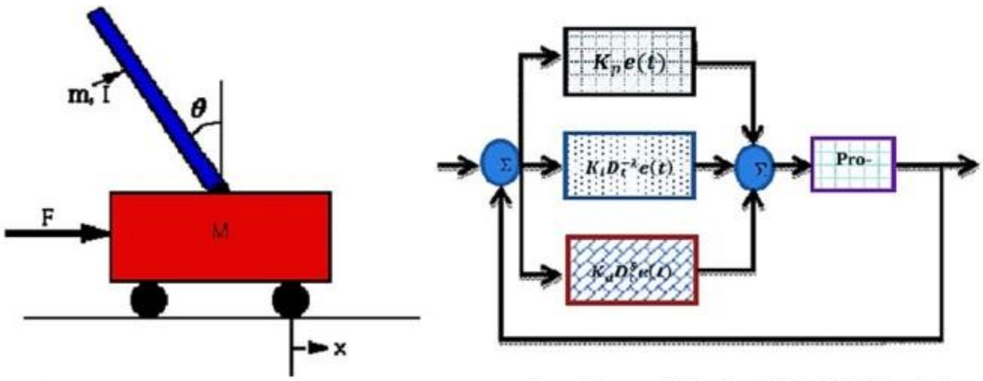
There are several definitions of a fractional derivative of order $\alpha > 0$ that we refer the interested reader to see [46] for more details about fractional operators but for the easy Laplace transformation of Caputo's derivative this derivative is widely used in the problems of control theory. The application of fractional problems can be found for system modeling in engineering and physics. These problems arise in many fields of engineering such as the overhead crane handling the cargo depicted in Fig. 1(a). The cargo is normally suspended on the cable by a hook. If the hook mass is included, then the physical model of the crane system can be viewed as a moving double-pendulum. Design of robust nonlinear controllers based on both conventional and hierarchical sliding mode techniques for double pendulum overhead crane systems [42, 43] (see Fig. 1(b)). Fractional calculus can provide novel and higher performance extension for designing fractional order PID controller [44] (see Fig. 2).

The paper is follow by: Section 2 that we introduce RBFs and interpolation by them. Also, we discuss the error estimate of the applied method in this section. Numerical experiments are carried out in Section 3, which will be used to verify the theoretical results obtained in the last section. Finally, we conclude the article in Section 4.



(a) Overhead crane system (b) Physical model

Fig. 1: Double-pendulum overhead crane system.



The structure of an Inverted Pendulum closed loop control system With a FOPID controller

Fig. 2: Closed loop control system With a PID controller.

II. RBF CALLOCATION METHOD FOR SOLVING FDDEs AND ERROR ESTIMATE

The industrial applications exploiting the expertise of systems and control engineering has a significantly wide spectrum covering the open-loop characterization of processes to develop high-performance feedback mechanisms, tuning of controllers via expert knowledge and computer tools, adaptive and selftuning mechanisms taking care of the changes in the process dynamics, nonlinear, robust and optimal control policies to meet a predefined set of performance criteria, fault tolerant schemes, and safety critical applications. The ultimate expectation in all such branches of industry is to maintain the production, while keeping the cost at minimum possible level.

Looking at the sectors and their needs, in the sequel, we propose a numerical computational approach based on RBF collocation method to directly solve the FDDEs. At first, we introduced and utilized the RBFs to interpolate a function in any dimensions. A function $\phi: \mathbb{R}^d \to \mathbb{R}$ is called to be radial if there exists a univariate function $\varphi: [0,\infty) \to \mathbb{R}$ such that $\phi(x) = \varphi(r)$, where $r = \| x \|$ and $\|.\|$ is usually Euclidean distance on $\mathbb{R}^d$, although other distance functions are also possible.

Definition 2 *Commonly used types of RBFs include the following forms in which $r = \| x - x_i \|$ and the shape parameter ε controls their flatness [45, 46]:*

- Piecewise Smooth:

$\phi(r) = r^3$, Cubic RBF

$\phi(r) = r^5$, Quintic RBF

$\phi(r) = r^2 log(r)$, Thin Plate spline (TPS) RBF

$\phi(r) = (1-r)^m + p(r)$, Wendland functions where p is a polynomial.

- Infinitely Smooth:

$\phi(r) = \sqrt{1+(\varepsilon r)^2}$, Multiquadric (MQ) RBF

$\phi(r) = \frac{1}{1+(\varepsilon r)^2}$, Inverse Quadratic (IQ) RBF

$\phi(r) = e^{-(\varepsilon r)^2}$, Gaussian RBF.

Next, we proceed by discussing on RBFs approximation.

Definition 3 *Given a global radial function $\phi(r)$, $r \geq 0$, scattered points $x_1, x_2, \cdots, x_N$ selected on the domain $\Omega \subset \mathbb{R}^d$ and data $f_i = f(x_i)$, $i = 1,2,\cdots,N$. Then, the basic* *interpolating RBF approximation for function $s(x)$ at an arbitrary point $x \in \Omega$ takes the form:*

$$s(x) = \sum_{i=1}^{N} \lambda_i \phi(\| x - x_i \|) + \psi(x), \tag{4}$$

where λ_i are the set of unknown RBF coefficients to be determined by the interpolation conditions $s(x_i) = f_i$, $i = 1,2,\cdots,N$ and ψ is a polynomial. The input variable x and points x_i can be either scalars or vectors. As well, in a similar representation as Eq. (4), for any linear fractional differential operator D^α, $D^\alpha s(x)$ can be approximated by:

$$D^\alpha s(x) = \sum_{i=1}^{N} \lambda_i D^\alpha \phi(\| x - x_i \|) + D^\alpha \psi(x) \tag{5}$$

Eq. (4) can be written without the additional polynomial ψ. In that case, ϕ must be strictly positive definite to guarantee the solvability of the resulting system (e.g., Gaussian or inverse multiquadrics). However, ψ is usually required when ϕ is conditionally positive definite, i.e., when ϕ has a polynomial growth toward infinity [46, 47]. Obtaining a closed form analytic expression for the fractional derivative of a radial function may lead to a challenge. Accordingly, Mohammadi and Schaback in [48], provide the required formulas for the fractional derivatives of RBFs in which allow us to use high order numerical methods for solving fractional problems such as FDDEs.

Now, we briefly introduce the RBFs collocation method. Consider the following boundary value problem when $\Omega \subset \mathbb{R}^d$:

$$Lu = f \quad in\, \Omega \tag{6}$$

$$u = g \quad on\, \partial\Omega \tag{7}$$

where L is a linear differential operator and d is the dimension of the problem. We distinguish in our notation centers of RBFs $X = \{x_1, \dots, x_N\}$ and the collocation points $\Xi = \{\alpha_1, \dots, \alpha_N\}$. Then we have the approximate solution of (2.3) and (2.4) in the form:

$$\tilde{u}(x) = \sum_{i=1}^{N} \lambda_i \phi(\| x - x_i \|), \tag{8}$$

where λ_i, $i = 1,2,\cdots,N$, are unknown coefficients that determined by collocation, ϕ is a RBF, $\|.\|$ is the Euclidean norm and x_i is the centers of the RBFs.

Now, let Ξ divided into two subsets. One subset contains N_I centers, Ξ_1, where Eq. (6) is enforced and the other subset contains N_B centers, Ξ_2, where boundary conditions are enforced. The collocation matrix that is obtained by matching the differential equation and the boundary condition at the collocation points has the following form:

$$A = \begin{bmatrix} A_I \\ A_B \end{bmatrix},$$

in which, $A_I = L\phi(\| \alpha - x_j \|)_{\alpha=\alpha_i}$, $\alpha_i \in \Xi_1$, $x_j \in X$, and $A_B = L\phi(\| \alpha - x_j \|)_{\alpha=\alpha_i}$, $\alpha_i \in \Xi_2$, $x_j \in X$. The unknown coefficients λ_i are determined by solving the linear system $A\lambda = F$, where F is a vector included $f(\alpha_i)$, $\alpha_i \in \Xi_1$, and $g(\alpha_i)$, $\alpha_i \in \Xi_2$.

A. APPLICATION OF RBF COLLOCATION METHOD

In this section, we will develop the RBF collocation method for solving problem (1). For this purpose, we rewrite problem (1) as the following form:

$$\begin{aligned} &D^\alpha u(t) - a(t)u(t) - b(t)u(t-\tau) = f_1(t); \quad 0 < t \le T, \\ &u(t) = 0, \quad -\tau \le t \le 0, \end{aligned} \tag{9}$$

where $u(t) = x(t) - w(t)$ is a new unknown function that

$$w(t) = \begin{cases} \varphi(t), & t \in [-\tau, 0] \\ 0, & t \in (0, T] \end{cases}$$

and

$$f_1(t) = \begin{cases} f(t) + b(t)\varphi(t-\tau), & t \in [0, \tau] \\ f(t), & t \in (\tau, T]. \end{cases}$$

The unknown functions $u(t)$ and $u(t-\tau)$ are approximated using N RBFs as following:

$$u(t) \simeq \sum_{i=1}^{N} \lambda_i \phi(\| t - t_i \|) = \sum_{i=1}^{N} \lambda_i \phi_i(t) \tag{10}$$

$$u(t-\tau) \simeq \sum_{i=1}^{N} \lambda_i \phi(\| t - \tau - t_i \|) = \begin{cases} 0, & 0 \le t \le \tau \\ \sum_{i=1}^{N} \lambda_i \phi_i(t-\tau), & \tau \le t \le T. \end{cases} \tag{11}$$

By substituting (10)-(11) in (9), we have:

$$\begin{aligned} &\sum_{i=1}^{N} \lambda_i D^\alpha \phi_i(t) - a(t) \sum_{i=1}^{N} \lambda_i \phi_i(t) - b(t) \sum_{i=1}^{N} \lambda_i \phi_i(t-\tau) := f_1(t) \quad 0 < t \le T, \\ &\sum_{i=1}^{N} \lambda_i \phi_i(t) = 0, \quad -\tau \le t \le 0, \end{aligned} \tag{12}$$

Rewrite (12) for collocation nodes $\{t_k\}_{k=1}^{N}$, yields a system of N linear equations with N unknowns as follow:

$$\begin{aligned} &\sum_{i=1}^{N} \lambda_i \left\{ D^\alpha \phi_i(t_k) - a(t_k)\phi_i(t_k) - b(t_k)\phi_i(t_k - \tau) \right\} = f_1(t_k); \; 0 < t_k \\ &\sum_{i=1}^{N} \lambda_i \phi_i(t_k) = 0, \; -\tau \le t_k \le 0. \end{aligned} \tag{13}$$

One can write this system in matrix form as:

$$\begin{pmatrix} A \\ B \end{pmatrix} \Lambda = \begin{pmatrix} 0 \\ F \end{pmatrix}, \tag{14}$$

where A is the $1 \times N$ matrices with elements $A_{ij} = \phi_j(t_i)$, $j = 1,2,\cdots,N$, $i = 1$, and B is the $(N-1) \times N$ matrices with elements

$B_{ij} = D^{\alpha}\phi_j(t_{i+1}) - a(t_{i+1})\phi_j(t_{i+1}) - b(t_{i+1})\phi_j(t_{i+1} - \tau)$

furthermore, $\Lambda \in \mathbb{R}^N$ is the vector of unknown coefficients and the components of F are the data $f_1(t_j)$, $j = 2,3,\cdots,N$. Finally, by solving the above system of algebraic equations, we can find the unknown coefficients $\{\lambda_i\}_{i=1}^N$. Consequently $u(t)$ can be approximated by (10) and $x(t)$ is approximated from $x(t) = u(t) + w(t)$.

B. Error estimate

In this section, the error analysis of the applied method for solving FDDEs will be provided. Exponential convergence estimates for the RBFs interpolation provided from [46]. For simplicity, in the next, we will assume that $\Omega = [0,1]$. In this position for subsequent error analysis, we have to consider the following definition and theorems.

Definition 4 *A real-valued continuous even function ϕ is called positive definite on Ω if*

$$\sum_{j=1}^N \sum_{k=1}^N c_j c_k \phi(x_j - x_k) \geq 0, \qquad (15)$$

for different points $x_1, x_2, \cdots, x_N \in \Omega$, and $c = [c_1, c_2, \cdots, c_N]^T \in \mathbb{R}^N$. The function ϕ is called strictly positive definite on Ω if the only vector $c = 0$ that turns (2.12) into an equality is the zero vector [46].

Theorem 1 *[46] Suppose that ϕ is positive definite RBF with infinite smoothness. Let $s(x)$ be the interpolate to f by using RBF approximation ϕ in (4). Then, there is a positive constant C, independent of X, such that*

$$\| f(x) - s(x) \|_{\infty} \leq C h_X |f(x)|_{\aleph_\phi(\Omega)}, \qquad x \in \Omega, \qquad (16)$$

where $\aleph_\phi(\Omega)$ is the so called real native Hilbert space of ϕ on Ω (the idea of the native spaces introduced in [49]) and

$$h_X = \sup_{x\in\Omega} \min_{x_i \in X} \| x - x_i \|, \qquad (17)$$

is the fill distance of a set of points $X = \{x_1, x_2, \cdots, x_N\}$ on Ω.

Therefore, the infinitely smooth RBFs such as Gaussians and (inverse) multiquadrics have a high convergence rates for local interpolation RBFs [49]. As a result of Theorem 1, the error for the RBF interpolation of delay function in (1) is similar to (16) and for the fractional derivative of $f \in \aleph_\phi(\Omega) \cap W_\infty^m(\Omega)$ defined by (5) we have [50]:

$$\| D^\alpha f(x) - D^\alpha s(x) \|_\infty \leq C N^{-1} \left(\sup_{x\in\Omega} \left| f(x) \right|_{m, L_2(\Omega)} + \left| f(x) \right|_{\aleph_\phi(\Omega)} \right), \qquad (18)$$

where, ϕ is strictly positive definite and W_∞^m is the Sobolev space.

Let us suppose that $E(t) = u(t) - \hat{u}(t)$ is the error function where $u(t)$ is the exact solution of the FDE (9) and $\hat{u}(t)$ is that the approximate solution of this equation that is given by Eq. (10) . Therefore,

$$D^\alpha \hat{u}(t) - a(t)\hat{u}(t) - b(t)\hat{u}(t-\tau) - f_1(t) = R_1(t);\ 0 < t \leq T,$$
$$\hat{u}(t) = R_2(t), \quad -\tau \leq t \leq 0. \qquad (19)$$

Now, by subtracting (9) from (19), we have:

$$D^\alpha (u - \hat{u})(t) - a(t)(u - \hat{u})(t) - b(t)(u - \hat{u})(t - \tau) = -R_1(t);$$
$$u(t) - \hat{u}(t) = -R_2(t), \quad -\tau \leq t \leq 0. \qquad (20)$$

Now, the error function $E(t)$ is established by the following equation:

$$D^\alpha E(t) - a(t)E(t) - b(t)E(t-\tau) = -R_1(t); \quad 0 < t \leq T,$$
$$E(t) = -R_2(t), \quad -\tau \leq t \leq 0, \qquad (21)$$

in which, $R_1(t)$ and $R_2(t)$ are known functions in the collocation points. So, to find approximate error, we can follow the same method mentioned in the last section.

III. RESULT AND DISCUSSIONS

In this section, we use the presented numerical approach to solve two illustrative examples. Based on the pervious discussion, as $N \to \infty$, the error of the RBF interpolation is bounded. So, the convergence behavior of the method is confirmed by some numerical examples in this section. All the computations were carried out using the Matlab software. Here, we will consider the power RBF as $\phi(r) = r^n$ with $n = 3$. Here, the error between the exact solution $x(t)$ and the approximate solution $\tilde{x}(t)$, found using our method, namely the absolute error and computed as follows:

$$Error\{x(t), \tilde{x}(t)\} = \| x(t) - \tilde{x}(t) \|_\infty,$$
$$t \in [0, T] \qquad (22)$$

Example 1 As an applicable problem, we consider the problem of the effect of noise on light which is reflected from laser to mirror as follows*:*

$$D^\alpha x(t) = \frac{-1}{\epsilon} x(t) + \frac{1}{\epsilon} x(t) x(t-1),$$
$$0 < \alpha \leq 1, \quad t > 0,$$
$$x(t) = 0.9, \qquad -1 \leq t \leq 0.$$

Fig. 3 illustrates the optoelectronic device used by Saboureau et al. [51]. The feedback operates on the pump of the laser by using part of the output light which is injected into a photodetector connected to the pump. The delay of the feedback is controlled by changing the length of the optical path. Applying the RBF collocation method for this boundary problem leads to the Fig. 4.

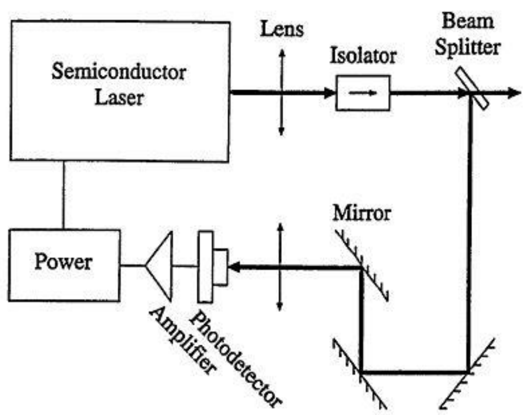


Fig. 3: Semiconductor laser subject to an optoelectronic feedback.

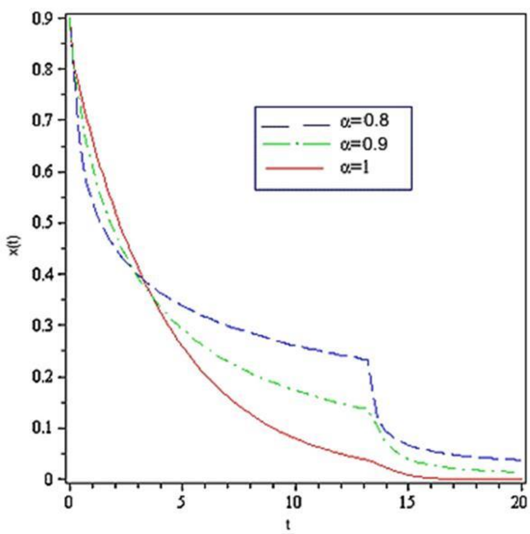


Fig.4: The numerical solution of Example (1) at different values of α with $\epsilon = 0.1$ and $N = 20$.

Example 2 *Consider the following FDDE:*

$$D^{\frac{1}{2}}x(t) = x(t-1) - x(t) + 2t - 1 + \frac{\Gamma(3)}{\Gamma\left(\frac{5}{2}\right)}t^{\frac{3}{2}}, \quad t > 0$$

$$x(t) = t^2, \qquad t \in [-1,0].$$

The exact solution in this case is $x(t) = t^2$ for $t \geq 0$. This problem was introduced by Morgado et. al. [12] in which the authors analyzed existence and uniqueness solutions for initial value problems of linear FDDEs. The authors in [5] used the shifted Jacobi polynomials for solving this problem. The method based on reproducing kernel space for approximated solution of this problem obtained by [50]. In Fig. 5, we plotted the exact solution and the approximated solution of $x(t)$ for $N = 15,25,30$. As expected, we can observe by increasing the value of N, the approximated solution converge to the exact values. The absolute errors of the presented method for this example, shown in Fig. 6 and Fig. 7. To evaluate the efficiency and performance of the RBF collocation method presented in Section 2 is better than those introduced in other literatures, a comparison is made between the absolute errors obtained by our method for different values of N with the best results that achieved by other researchers in Table 1. From the perspective of this table, our suggested approach is somewhat more effective by increasing N.

Example 3 *Consider the following FDDE:*

$$D^{0.3}x(t) = x(t-1) - x(t) + 1 - 3t + 3t^2 + \frac{2000t^{2.7}}{1071\Gamma(0.7)}$$

$$x(t) = t^3, \qquad t \in [-1,0],$$

The exact solution in this case is $x(t) = t^3$ for $t \geq 0$. The absolute errors of the presented method for this example, shown in Fig. 8. Also, Table 2, demonstrates the effect of various values of N for this example. Comparing the numerical results obtained in this table, reveals that the accuracy of the RBF collocation method is higher than the methods presented in [12, 52].

Table I: Absolute errors and CPU time(second) at different choices of N in [0,100] for Example (2).

N	Ref. [12]	This study	CPU time
1000	0.0491843	0.0785	1.864828
2000	0.0276172	0.0221	7.398307
4000	0.0146507	0.0063	30.177423
8000	0.0075649	0.0018	129.507482

III. CONCLUSIONS

The method of RBFs is a method of scattered data interpolation that has many application in different fields. In spite of easy implementation of other high-order methods and finite difference schemes for solving a problem of fractional order derivatives, the challenge of these methods is their limited accuracy, locality, complexity and high cost of computing in discretization of the fractional terms, which suggest that global scheme such as RBFs that are more accurate way for discretizing fractional calculus and would allow us to remove the ill-conditioning of the system of discrete equations. So, the proposed approach, either employ any global RBFs for interpolating technique or uses arbitrary points for discretization, offers a very flexible framework for solving FDDEs. From the presented results in the last section, it could be concluded that the results derived by the RBFs collocation method have a good agreement with the true solution given by average approximation.

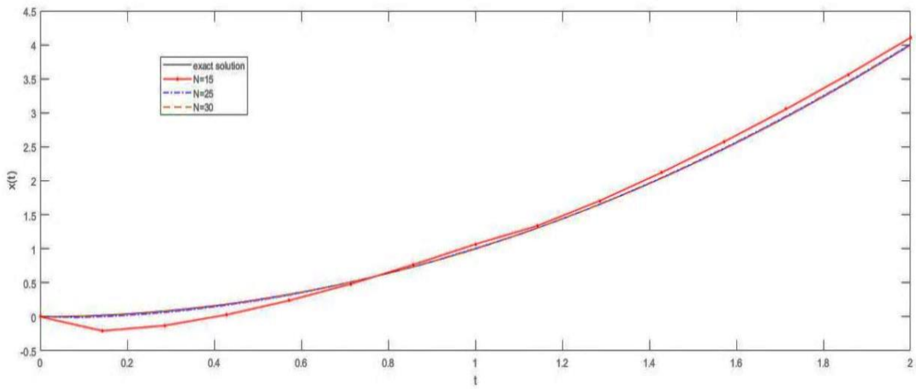


Fig. 5: Exact and approximate values of $x(t)$ at different values of N in $[0,2]$ for Example (2).

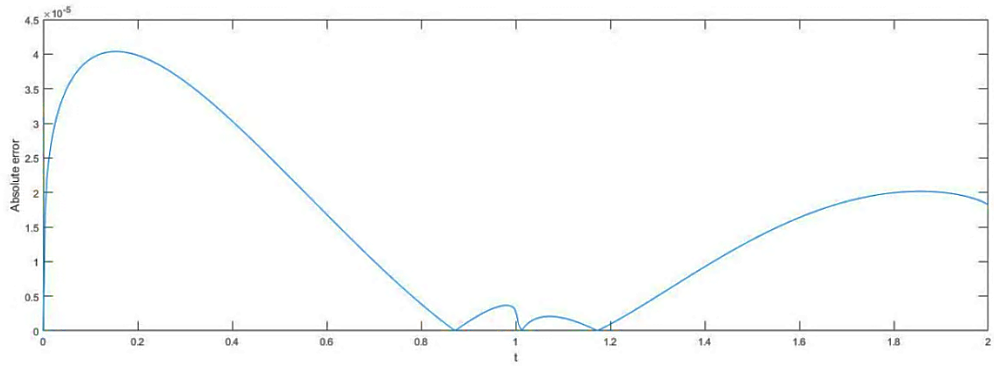


Fig. 6: Absolute error of $x(t)$ with $N = 500$ in $[0,2]$ for Example (2).

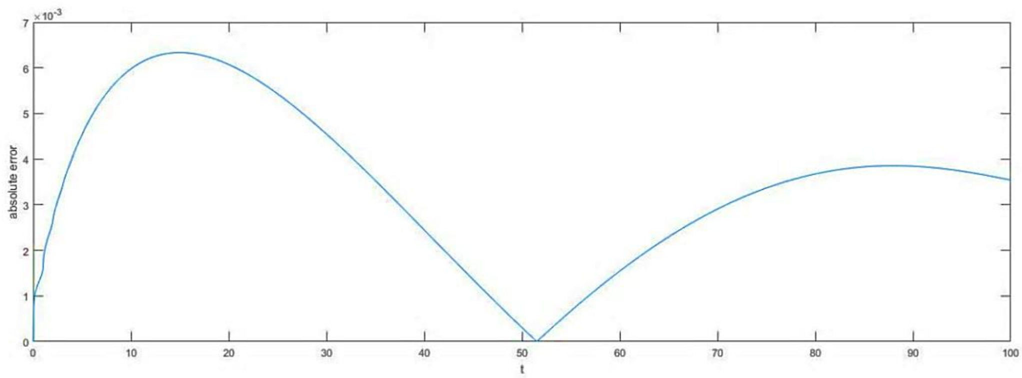


Fig. 7: Absolute error of $x(t)$ with $N = 4000$ in $[0,100]$ for Example (2).

Table II: Absolute error and CPU time (second) at different choices of N in $[0,100]$ for Example (3).

N	Ref. [12]	Ref. [52]	This study	CPU time
1000	0.0710508	0.00001	$3.0268e-09$	1.855609
2000	0.0411543	$--$	$6.4028e-09$	7.475098
4000	0.0219612	$--$	$1.7899e-08$	31.21183
8000	0.0113125	$--$	$2.5844e-08$	132.473657

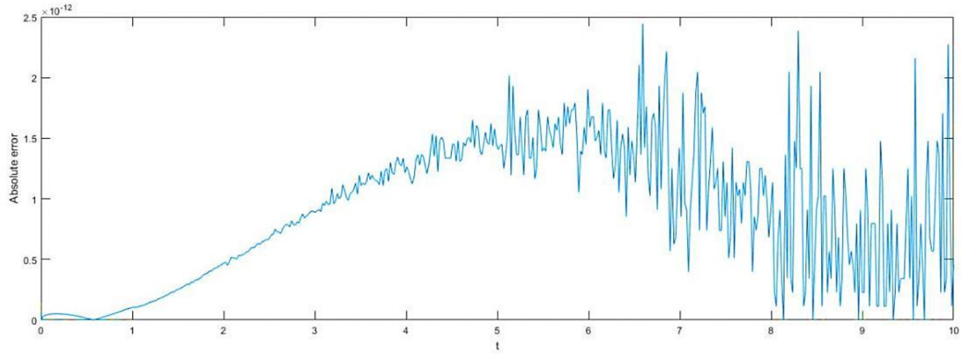


Fig. 8: Absolute error of $x(t)$ with $N = 500$ in $[0,10]$ for example (3).

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Fahimeh Akhavan-Ghassazadeh received the .S. and M. S. degrees and also her Ph.D. in applied mathematics from Ferdowsi University of Mashhad, Iran, in 2006, 2008 and 2018, respectively. She is an assistant professor at the University of Gonabad. Her main research interests are the radial basis functions, fractional differential equations, Spectral method, and the delay differential equations.

Samaneh Soradi-Zeid received his B.S. degree in Apllied Mathematics fom National University of Birjand, Iran, in 2010, and his M.S. degree in Operation Research from Sharif University of Tehran, Iran, in 2012 and his Ph.D. degree in Optimal Control and Optimization from Ferdowsi University of Mashhad, Iran, in 2017. From September 2018 she works as an Assistant Professor at the Faculty of Industry and Mining (khash), University of Sistan and Baluchestan, Zahedan. Iran. Her current research interests include Fractional Optimal control, Delay differential Equations, Nonlinear Optimization, Collocation and Spectral Methods.

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