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A Novel Method for Optimal Control of Piecewise Affine Systems Using Semi-Definite Programming

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ABSTRACT

In this paper, a novel optimal control design method by discontinuous quadratic Lyapunov function and continuous quadratic Lyapunov function for 2-dimensional piecewise affine systems via semi-definite programming with LMI constraints is proposed. At the first, an upper bound for a quadratic cost function for a stable closed-system is obtained. Then after, considering a state-feedback control approach, not only sufficient conditions for the stability of the closed-loop system but also the upper bound of the cost function are obtained. The optimization problem is formulated as a semi-definite programming with bilinear constraints (BMI). Some variables in BMIs are searched by genetic algorithm, so the bilinear constraints are converted to linear constraints and the controller coefficients are calculated. The effectiveness of the proposed method is verified by numerical examples.

Article Info

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I. Introduction

Hybrid systems are dynamical systems with continuous-time dynamics and discrete events. These systems are applied for modeling various real-world applications. Piecewise affine (PWA) systems are a subset of hybrid systems and their equivalence with some other classes of the hybrid systems are shown in [1]. As we know, many of nonlinearities such as saturation are either inherently modeled in the form of PWA systems or approximated as a PWA system [2]. Accordingly, the class of PWA systems is an important tool as well as a starting point for modeling and analysis of nonlinear systems. These systems are defined by partitioning the state-space into a finite number of polyhedral regions and associating a different affine dynamic model to each region. Power electronics, process control and a wide range of nonlinear systems in engineering are some of the attractive applications of PWA systems in recent years, for instance, see, [3-5] The excellence advantage of PWA systems is that the stability and performance analysis of such systems can be formulated as a convex problem which is easily solved by numerical methods. Controllability and observability of PWA systems discussed in [6]. In[7] , stability analysis is expressed in the form of linear matrix inequalities (LMIs). In control theory, different approaches are proposed to define the optimal control law. Different algorithms for optimal control of continuous-time hybrid systems are compared in [8]. The necessary conditions for optimal control law in hybrid systems using dynamic programming and maximum principle can be found in [9]. To approximate optimal control law, [10] uses dynamic convex programming. Recently,

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writers investigated the stabilization problem of an autonomous linear time invariant switch systems [11] , [12] discusses the optimal control of hybrid systems in a finite-time interval and gives the necessary conditions for optimization using the maximum principle, designing controller for wireless sensor networks by Linear Matrix Inequality are given in [13] .In [14-16] optimal control schemes of switched systems are reported, moreover, [17-19] study the problem of optimal control in switched affine systems. One of the most challenging issues over recent years was controller design and synthesis for PWA systems. Quadratic control and calculation of L^2 gain for these systems are introduced in [20] and [21],respectively. The reference [22] has discussed the issue of linear quadratic regulator (LQR) of finite-time for PWA systems using the measure theory. The optimal control for sampled-data PWA systems is studied in [23] .For discrete-time PWA systems, an optimal controller is designed in [24]. In [25], the author proposes some theorems on calculation of upper limits regarding optimal control for PWA systems, however no controller is designed. The issue of optimal control for PWA systems with uncertainty using output-feedback is studied in [26] and then based on numerical algorithms the controller is obtained by solving a series of LMIs. However, there is not much research done in the field of PWA systems optimal control and the majority of the studies are related to [27] programming, optimal control approaches for linear time invariant and time variant systems are reported in [28] and [29], respectively. In [30], nonlinear predictor feedback for input-affine systems with distributed input delays is considered. In [31], stabilizing switching laws for mixed switched affine systems is designed. Optimal LQ-type switched control design for a class of linear systems with piecewise constant inputs is introduced in [32]. On the other hand, optimal hybrid perimeter and switching control schemes for urban traffic networks is suggested in . Control of piecewise affine networked control system is referred in [33]. In [34] piecewise affine system identification of a hydraulic wind power transfer system is reported and in [35] switching rule design for affine switched systems with guaranteed cost and uncertain equilibrium condition is well established.

In this paper, assuming state feedback control, the upper bound for the related linear quadratic regulator cost function is calculated and the optimal controller design problem based on discontinuous quadratic Lyapunov function and continuous quadratic Lyapunov which leads to solving a semi-definite programming problem with bilinear constraint is converted into a semi-definite programming problem with linear constraints using genetic algorithm (GA).

The rest of this paper is organized as follows. In Section II necessary definitions and backgrounds are given. Stability analysis based on discontinuous quadratic Lyapunov function and continuous quadratic Lyapunov is presented in section III. The upper bound for the related cost function is calculated in section IV. The optimal controller design and numerical examples are described in sections V and VI, respectively. Finally, conclusions are reported in section VII.

We have designed optimal control problem for piecewise affine systems based on discontinuous quadratic Lyapunov function previously in [36] . In this paper we design optimal control based on discontinuous quadratic Lyapunov function and continuous quadratic Lyapunov and compare these optimal controller.

II. Definitions

In this section, necessary definitions and backgrounds for studying the next sections are described.

Linear matrix inequality (LMI)

Linear matrix inequality is defined as follows [37]

$$F_0 + x_1F_1 + \cdots . + x_mF_m \geq 0 \tag{1}$$

In which $x \epsilon R^n$ are the problem variables and matrices $F_i = F_i^T$ are matrices related to the problem. The inequality sign is denoted as positive semi-definite when $A \geq 0$ the matrix is and when $A > 0$ the matrix is a positive definite.

Inner product

We show the space of all $n \times n$ real matrices using s^n. This space is equipped with an inner product in form of: [38]

$$< A < B >_{s^n} = Tr(A^TB) = \sum_{i=j=1}^{n} A_{ij}B_{ij} \tag{2}$$

Semi-definite programming

The purpose of solving the semi-definite programming problem is to minimize the inner product $< C, X >= Tr(C, X)$ so that both C and X matrices are symmetric $n \times n$ matrices, with C being the fixed matrix and X the variable. Tr rep resents the sum of the diagonal elements of the matrix. The problem's constraints are divided into two categories; the first one is linear constraints in form of [38]

$$Tr(A_iX) = b_i \quad i = 1,2,3, \dots , n \tag{3}$$

In which A_i are symmetric matrices and b_i are scalars. The second category of constraints is convex and non-linear constraints like $X \geq 0$. With this background discussed, now we can define the semi-definite programming problem as:

$$(P): P^* = \inf\{Tr(CX): Tr(A_iX) = b_i , X \geq 0\} \tag{4}$$

The dual of the above problem is defined as:

$$(D): d^* = \sup\left\{b^TY: \sum_{i=1}^{m} Y_iA_i + S = C, S \geq 0, Y \in R^m\right\} \tag{5}$$

Piecewise affine (PWA) systems

The mathematical description of PWA class system in general is [25]

$$\begin{cases} \dot{x} = a_i + A_i x + B_i u \\ y = c_i + C_i x + D_i u \end{cases} \quad \text{for } x \epsilon X_i \tag{6}$$

In which X_i are the corresponding regions and their collection is partitioned by the state space.

$$X_i = \{x \epsilon R^2, E_i x \geq e_i\}\ i \epsilon I \tag{7}$$

In (7) E_i and e_i are respectively a matrix and a vector, with constant value and appropriate size.
It's worth noting that if $B_i = 0$, the system (6) turns into a piecewise linear system as follows:

$$\dot{x} = a_i + A_i x \quad \text{for } x \epsilon X_i \tag{8}$$

III. Stability Analysis

Stability of a system has a variety of definitions. In this paper, we call a system stable if it has global Lyapunov stability. The methods we use in order to evaluate the stability are discontinuous piecewise quadratic Lyapunov function [39] and continuous piecewise quadratic Lyapunov function[7]. For this, first we discuss the necessary background:
Suppose the PWA system defined in the previous section, is continuous in its boundary, so for $\mathbf{x \epsilon \overline{X_i} \cap \overline{X_j}}$ we have: [39]

$$A_i x + a_i = A_j x + a_j \tag{9}$$

If region X is partitioned as polyhedron in this case; each sub-region can be described as equation (10)

$$\overline{X_i} = \{x \epsilon R^2 : E_i x \geq e_i\}, i \epsilon I \tag{10}$$

Where E_i and e_i are respectively a matrix and a vector, with constant value and appropriate size. A parametric description of the boundary between two regions $\overline{X_i}$ and $\overline{X_j}$ where in $\overline{X_i} \cap \overline{X_j} \neq \emptyset$, can be described as

$$\overline{X_i} \cap \overline{X_j} \subseteq \{x \mid x = F_{ij} s + f_{ij}, s \epsilon R\} \tag{11}$$

$$\overline{X_i} \cap \overline{X_j} \subseteq \{x \mid \bar{x} = \overline{F_{ij}} \bar{s}, \bar{s} = \begin{pmatrix} s \\ 1 \end{pmatrix}, s \epsilon R\} \tag{12}$$

$$\overline{E_i} = [E_i \quad -e_i] \tag{13}$$

$$\overline{F_{ij}} = \begin{pmatrix} F_{ij} & f_{ij} \\ 0 & 1 \end{pmatrix} \tag{14}$$

In this relation if $F_{ij} \neq 0$ the border is a part of a line and if $F_{ij} = 0$ the border is a point. To be more specific, you can refer [38].
For the two adjacent regions X_i and X_j, we assume that $F_{ij} \neq 0$ (the border is part of a line) one can define vector $\overline{C_{ij}} = [C_{ij} \ c_{ij}]$ and hyper plane line as $s_{ij} = \{x \mid \overline{C_{ij}} \bar{x} = 0\}$. which C_{ij}, is the normal vector of s_{ij} (perpendicular to s_{ij}), with the direction from X_i to X_j, so that $\bar{X}_i \cap \bar{X}_j \subseteq s_{ij}$ is satisfied.
According to what was said, now we can propose the theorem regarding the stability of PWA systems. Consider the resulting Lyapunov function as the following relation:

$$V(x) = V_i(x) = \bar{x}^T \overline{P_i}, \overline{p_i} = \overline{p_i}^T \epsilon R^{3\times 3} \tag{15}$$

$$r_i \epsilon R, q_i \epsilon R^2, p_i \epsilon R^{2\times 2} \overline{P_i} = \begin{pmatrix} p_i & q_i \\ q_i^T & r_i \end{pmatrix} \tag{16}$$

Theorem1: [39]

Suppose $\overline{U_i}$ and $\overline{W_j}$ are unknown matrices with non-negative elements and appropriate dimensions and (k=1,2) $\overline{W_{ij}}^k$ are unknown vectors with appropriate dimensions and non-negative elements and (i∈I) $\overline{p_i}$ $\epsilon R^{3\times 3}$ is a symmetric matrix, then define the following variables :

$$\overline{H_{ij}} = \overline{E_i}^T \overline{\omega_{ij}}^1 \overline{C_{ij}} \overline{A_i} + \overline{E_j}^T \overline{\omega_{ij}}^2 \overline{C_{ij}} \overline{A_j}$$
$$\overline{M_i} = \overline{E}_i^T \overline{W}_i \overline{E}_i \tag{17}$$

If there is a choice between $\overline{P_i}$ and $\overline{U_i}$ and $\overline{W_i}$ matrices and (k=1,2) $\overline{W_{ij}}^k$ vectors that satisfy the following restrictions, then for system defined by equation (8) all the trajectory starting at X will exponentially converge to origin.

$$\overline{P_i} = \begin{pmatrix} p_i & 0 \\ 0 & o \end{pmatrix} > 0 \, for \ i \epsilon I_0 \tag{18}$$

$$\overline{\overline{P_i}} - \overline{L_i} > 0 \ for \ i \epsilon I_1 \ \ i \notin I_0 \tag{19}$$

$$(I_n \ 0)(\overline{P_i} - \overline{L_i})\begin{pmatrix} I_n \\ 0 \end{pmatrix} > 0 \quad \forall i \in I_0 \tag{20}$$

$$\overline{A_i}^T \overline{P_i} + \overline{P_i A_i} + \overline{M_i} < 0 \quad \forall i \in I, i \notin I_0 \tag{21}$$

$$(I_n \ 0)\left(\overline{A_i}^T \overline{P_i} + \overline{P_i A_i} + \overline{M_i}\right)\begin{pmatrix} I_n \\ 0 \end{pmatrix} < 0 \quad \forall i \in I_0 \tag{22}$$

$$\overline{F_{ij}}^T (\overline{P_i} - \overline{P_j}) \overline{F_{ij}} = \overline{F_{ij}}^T \left(\overline{H_{ij}} + \overline{H_{ij}}^T\right) \overline{F_{ij}} \forall i \in I, j \in N_i, \text{where } F_{ij} \neq 0 \tag{23}$$

$$\text{Where } N_i = \{k \in I, k \neq i, \overline{X_i} \cap \overline{X_j} \neq \emptyset\} \tag{24}$$

Theorem2:[7]

Consider symmetric matrices T and U_i and W_i such that U_i and W_i have nonnegative entries, while

$$P_i = F_i^T T F_i, i \epsilon I_0$$
$$\overline{P_i} = \bar{F}_i^T T \bar{F}_i, i \epsilon I_1 \tag{25}$$

Satisfy

$$\begin{cases} 0 > A_i^T P_i + P_i A_i + E_i^T U_i E_i \ i \epsilon I_0 \\ \quad 0 < P_i - E_i^T U_i E_i \end{cases} \tag{26}$$

$$\begin{cases} 0 > \bar{A}_i^T \overline{P_i} + \overline{P_i A_i} + \bar{E}_i^T U_i \bar{E}_i \ i \epsilon I_1 \\ \quad 0 < \bar{P}_i - \bar{E}_i^T U_i \bar{E}_i \end{cases} \tag{27}$$

Then every continuous piecewise C^1 trajectory $x(t)$ satisfying (8) for $t > 0$,tends to zero exponentially.

IV. CALCULATING UPPER BOUND

Theorem 3: [25]

For the system (8) if the conditions of theorem (1) are met and the inequality is established, then the upper bound for cost function $j = \int_0^\infty x(t)^T Q_i x(t) dt \quad Q_i > 0$ is calculated:

$$i \in I_0 \; P_i A_i + A_i^T P_i + Q_i + E_i^T \overline{W_i} E_i < 0 \quad (28)$$

$$i \in I_1 \; \overline{P_i A_i} + \overline{A_i}^T \overline{P_i} + \overline{Q_i} + \overline{E_i}^T \overline{W_i}\, \overline{E_i} < 0 \quad (29)$$

Proof: This proof is given from [25]. Suppose that $i \in I_1$ we prove theorem for $i \in I_1$, another proof is the same. By multiplying the said inequality by X from left and right and removing of non-negative terms. Then we take the integral of the expression that we desire in the interval $[0, \infty]$:

$$\bar{x}^T \overline{P_i A_i} \bar{x} + \bar{x}^T \overline{A_i}^T \overline{P_i} \bar{x} + \bar{x}^T \overline{Q_i} \bar{x} + \bar{x}^T \overline{E_i}^T \overline{W_i}\, \overline{E_i} \bar{x} < 0$$

$$\bar{x}^T \overline{P_i} \dot{\bar{x}} + \dot{\bar{x}}^T \overline{P_i} \bar{x} + \bar{x}^T \overline{Q}_i \bar{x} + \bar{x}^T \overline{E_i}^T \overline{W_i}\, \overline{E_i} \bar{x} < 0$$

$$\frac{d}{dt}\left(\bar{x}^T \overline{P_i} \bar{x}\right) + \bar{x}^T \overline{Q_i} \bar{x} + \bar{x}^T \overline{E_i}^T \overline{W_i}\, \overline{E_i} \bar{x} < 0$$

$$\left(\bar{x}^T \overline{E_i}^T \overline{W_i}\, \overline{E_i} \bar{x}\right) \geq 0 \Longrightarrow$$

$$\frac{d}{dt}\left(\bar{x}^T \overline{P_i} \bar{x}\right) + \bar{x}^T \overline{Q_i} \bar{x} \leq 0$$

$$\int_0^\infty \left[\frac{d}{dt}\left(\bar{x}^T \overline{P_i} \bar{x}\right) + \bar{x}^T \overline{Q_i} \bar{x}\right] dt \leq 0$$

$$\left[\bar{x}^T \overline{P_{i0}} \bar{x}\right]_0^\infty + j \leq 0$$

$$0 - \overline{x(0)}^T \overline{P_{i0}} \overline{x(0)} + j \leq 0$$

$$j \leq \overline{x(0)}^T \overline{P_{i0}} \overline{x(0)}$$

$$E(j) \leq E\left(tr\left(\overline{P_{i0}} \bar{x}_0 \bar{x}_0^T\right)\right) = \sum_{i \in I} \alpha_i tr\left(\overline{P_{i0}} \overline{L_i}\right) \quad (30)$$

$$L_i = \begin{cases} E(x_0 x_0^T) x_0 \in X_i, i \in I_0 \\ E(\bar{x}_0 \bar{x}_0^T) x_0 \in X_i, i \in I_1 \end{cases} \quad (31)$$

What is important for continuing this paper is (30) because we should use it for designing controller.

You may notice in the above inequalities that all of them are a series of LMIs in relation to the variables P_i and $(\overline{P_i})$ Therefore, stability conditions for a closed-loop system are a series of LMIs in relation to P_i and $(\overline{P_i})$ and they are convex optimization problems that can be solved using numerical methods. Note that because the cost function is dependent on the initial point and this point is an unknown random variable, we assume that it has a uniform distribution, so the dependency is eliminated. Operator E expresses the expected value and α_i represents the probability that x_0 belongs to area X_i. Since we considered the initial state as a uniform random variable, therefore the probability of α_i and the covariance matrix L_i can be determined using the desired area's information and the partitionX_i.

V.OPTIMAL CONTROL

In this section we describe the optimal controller design issues for PWA systems using the state feedback. We assume that the designated system balance point is the initial point. Consider the system described with equations (6), in this case assume that state feedback controller is $u(t) = K_i x(t)$. The closed-loop system takes the form below:

$$\begin{cases} \dot{x} = (A_i + B_i K_i) x(t) + a_i \\ \quad x(t_0) = x0 \end{cases} \quad (32)$$

We consider the cost function as:

$$j(x_0, u) = \int_0^\infty [x(t)^T Q_i x(t) + u(t)^T R_i u(t)] dt \quad (33)$$

With the consideration of the appropriate state feedback, the cost functions come in the form of:

$$j(x_0, u) = \int_0^\infty [x(t)^T (Q_i + K_i^T R_i K_i) x(t)] dt \quad (34)$$

$$j(x_0, u) = \int_0^\infty (x(t)^T \; 1) \begin{pmatrix} Q_i + K_i^T R_i K_i & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ 1 \end{pmatrix} dt \quad (35)$$

$$j(x_0, u) = \int_0^\infty \overline{x(t)}^T \overline{Q_i}\, \overline{x(t)} dt \quad (36)$$

Using the notations of equation (35), the equation (29) gives:

$$\dot{\overline{x(t)}} = \begin{pmatrix} A_i + B_i K_i & a_i \\ 0 & 0 \end{pmatrix} \overline{x(t)} \quad (37)$$

$$\overline{A_i} = \begin{pmatrix} A_i + B_i K_i & a_i \\ 0 & 0 \end{pmatrix}, \overline{x(t)} = \begin{pmatrix} x(t) \\ 1 \end{pmatrix} \quad (38)$$

By applying the mentioned changes in the form of the equations, theorem 3 for the system (37) is rewritten as:

Theorem 4:

For the system (37) with (assuming that the system is stable) if the equations (39-40) are met, then the upper bound for the equation (36) is obtained:

$$i \in I_0 \; P_i (A_i + B_i K_i) + (A_i + B_i K_i)^T P_i + Q_i + K_i^T R_i K_i + E_i^T \overline{W_i} E_i < 0 \quad (39)$$

$$i \in I_1 \overline{P_i A_i} + \overline{A_i}^T \overline{P_i} + \overline{Q_i} + \overline{E_i}^T \overline{W_i E_i} < 0 \quad (40)$$

In this case we'll have: $j \leq \overline{x(0)}^T \overline{P_i} \overline{x(0)}$

Proof: To prove this theorem in theorem 3, we convert A_i to $A_i + B_i K_i$. Now we can merge theorems 1, theorem 2 and theorem 4 and generally express the result in terms of theorem 5 and theorem 6:

Theorem 5:

For the system defined by equations (37) if the following conditions are met, then the system for each respective system trajectory exponentially converges to the origin and

$j \le \overline{x(0)}^T \overline{P_i}\, \overline{x(0)}$

$$i \in I_0 \quad \overline{P_i} = \begin{pmatrix} p_i & 0 \\ 0 & o \end{pmatrix} > 0 \qquad (41)$$

$$\left(\overline{P_i} - \overline{L_i}\right) > 0 \qquad (42)$$

$$\left(\overline{A_i}^T \overline{P_i} + \overline{P_i A_i} + \overline{M_i}\right) < 0 \qquad (43)$$

$$\overline{A_i} = \begin{pmatrix} A_i + B_i K_i & 0 \\ 0 & 0 \end{pmatrix} \qquad (44)$$

$$P_i(A_i + B_i K_i) + (A_i + B_i K_i)^T P_i + Q_i + K_i^T R_i K_i + E_i^T \overline{W_i} E_i < 0 \qquad (45)$$

$$i \in I_1 \; \overline{P_i} - \overline{L_i} > 0 \qquad (46)$$

$$\overline{A_i}^T \overline{P_i} + \overline{P_i A_i} + \overline{M_i} < 0 \qquad (47)$$

$$\overline{P_i A_i} + \overline{A_i}^T \overline{P_i} + \overline{Q_i} + \overline{E_i}^T \overline{W_i}\, \overline{E_i} < 0 \qquad (48)$$

$$\overline{A_i} = \begin{pmatrix} A_i + B_i K_i & a_i \\ 0 & 0 \end{pmatrix} \qquad (49)$$

for $x \in X_i \cap X_j$ we have

$$\overline{F_{ij}}^T \left(\overline{P_i} - \overline{P_j}\right) \overline{F_{ij}} = \overline{F_{ij}}^T \left(\overline{H_{ij}} + \overline{H_{ij}}^T\right) \overline{F_{ij}} \forall i \in I, j \in N_i, \text{where } F_{ij} \ne 0 \qquad (50)$$

$$\text{Where } N_i = \{k \in I, k \ne i, \overline{X_i} \cap \overline{X_j} \ne \emptyset\} \qquad (51)$$

$$(A_i + B_i K_i)x(t) + a_i = (A_j + B_j K_j)x(t) + a_j \;\Rightarrow$$

$$\begin{pmatrix} A_i + B_i K_i & a_i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} A_j + B_j K_j & a_j \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix}$$

$$X_i \cap X_j \subseteq \{x | \bar{x} = \overline{F_{ij}} \bar{s}, \bar{s} \epsilon R, \bar{s} = \begin{pmatrix} s \\ 1 \end{pmatrix}\}$$

$$\overline{F_{ij}} = \begin{pmatrix} F_{ij} & f_{ij} \\ 0 & 1 \end{pmatrix} \Rightarrow$$

$$\overline{A_i F_{ij}} = \overline{A_j F_{ij}} \qquad (52)$$

$$j \le \overline{x(0)}^T \overline{P_{i0}}\, \overline{x(0)} \Rightarrow$$

$$E(j) \le E\left(\mathrm{tr}\left(\overline{P_{i0}} \bar{x}_0 \bar{x}_0^T\right)\right) = \sum_{i \in I} \alpha_i \mathrm{tr}(\overline{P_{i0}}\, \overline{L_i}) \qquad (53)$$

$$L_i = \begin{cases} E(x_0 x_0^T) \; x_0 \in X_i, i \in I_0 \\ E(\bar{x}_0 \bar{x}_0^T) \; x_0 \in X_i, i \in I_1 \end{cases} \qquad (54)$$

This theorem is based on discontinuous quadratic Lyapunov that is given in [39].

Theorem 6:

For the system defined by equations (37) if the following conditions are met, then the system for each respective system trajectory exponentially converges to the origin and $j \le \overline{x(0)}^T \overline{P_i}\, \overline{x(0)}$

$$\begin{cases} 0 > A_i^T P_i + P_i A_i + E_i^T U_i E_i \; i \epsilon I_0 \\ 0 < P_i - E_i^T U_i E_i \\ P_i(A_i + B_i K_i) + (A_i + B_i K_i)^T P_i + Q_i + K_i^T R_i K_i + E_i^T \overline{W_i} E_i < 0 \end{cases} \qquad (55)$$

$$\begin{cases} 0 > \bar{A}_i^T \bar{P}_i + \overline{P_i A_i} + \bar{E}_i^T U_i \bar{E}_i \; i \epsilon I_1 \\ 0 < \bar{P}_i - \bar{E}_i^T U_i \bar{E}_i \\ \overline{P_i A_i} + \overline{A_i}^T \overline{P_i} + \overline{Q_i} + \overline{E_i}^T \overline{W_i E_i} < 0 \end{cases} \qquad (56)$$

This theorem is based on continuous quadratic Lyapunov that is given in [7].

Finally, for optimal control design, the coefficient K_i must be calculated. To calculate these coefficients we consider a controller that minimizes the upper bound of the cost function$\sum_{i \in I} \alpha_i \mathrm{tr}(\overline{P_{i0}}\, \overline{L_i})$. As we minimize the upper bound, the cost function will also be minimized; therefore, the desired optimization problem will actually lead to the design of the controller. In order to design optimal control we design optimal control based on theorem 5 and theorem 6 .we give these two optimal control in the form of optimization problem 1 and optimization problem 2.

Optimization problem 1(based on discontinuous quadratic Lyapunov function)

$$\min \left(\sum_{i \in I} \alpha_i \mathrm{tr}(\overline{P_{i0}}\, \overline{L_i})\right) \quad \text{subject to} \begin{cases} K_i \epsilon K \\ (41) - (52) \end{cases} \qquad (57)$$

Optimization problem 2(based on continuous quadratic Lyapunov function)

$$\min \left(\sum_{i \in I} \alpha_i \mathrm{tr}(\overline{P_{i0}}\, \overline{L_i})\right) \quad \text{subject to} \begin{cases} K_i \epsilon K \\ (55) - (56) \end{cases} \qquad (58)$$

As you can see, these two optimization problems are actually a semi-definite programming problem with LMI and BMI constraints such as (48) and (56) in fact they are two BMI problem, the references [32] use numerical V-K algorithms to solve problems with BMI constraints, but numerical algorithms V-K doesn't have a good convergence and is trapped in local minimum. Note that our desired functions are not dependent on the variables K_i and these variables can be seen in our objective function. Now if K_i is given, then our optimization problems turn into a semi-definite programming problem. GA is a comprehensive solution for high dimensional problems. Suppose that the set $\bar{K}$, is the set of all acceptable controllers for the controller coefficients K_i, if we find a way to calculate this coefficients, we have managed to design an optimal controller and the minimum value of cost function can be calculated. We have to calculate the controller coefficients using GA note that we don't use GA for minimizing the cost function. In order to find these coefficients using the mentioned method, we ascribe each chromosome in GA to a corresponding controller coefficientK_i. In which case, the non convex optimization problem turns into a semi-definite programming problem. Assuming that the controller coefficients K_i are known, we can calculate the fitness function in each chromosome.

If k is ascribed to each chromosome, then the fitness function will be defined as:

$$Fitness = \frac{1}{\bar{J}} \tag{59}$$

In which $\bar{J}$ is the minimum cost function and can be easily calculated using CVX toolbox. Note that if $\bar{J}$ corresponds to an infeasible chromosome, then the minimum value of the cost function will be infinite and its fitness function will be zero, this will stop the corresponding chromosome with impossible answer from generating next generation of children.
The algorithm routine used for two optimization problems are summarized as follows:
Step1: Initialize value of parameters related to the GA. (Specify the population size, percentage of crossover and percentage of mutation.)
Step2: Calculate the fitness function specified by the equation (59) for each chromosome or solution.
Step3: Using the calculated fitness function in step 2 and spinning the roulette wheel select one chromosome. Then, Use the given values such as percentage of crossover and percentage of mutation to complete crossover and mutation.
Step4: If the algorithm termination conditions are met, extract the results, otherwise, go to step 1.
GA is a efficient algorithm for searching the best solution. You can follow this subject in the reference [40]
As you can see, in two optimization problems values α_i and $\overline{L_i}$ are unknown, in order to treat this ambiguity we presume that initial state x_0 belongs to all sub-regions $X_i$with equal probability, therefore if n is the number of sub-regions, then two optimization problems takes the following form:

$$\min\left(\sum_{i\in I}\frac{1}{n}\mathrm{tr}(\overline{P_{i0}}\,\overline{L_i})\right) \tag{60}$$

Note that for solving optimization problem 1, although L_i exists in the optimization problem, it won't affect the answer since $E(\bar{x}_0\bar{x}_0{}^T)$is fixed. Also, we can easily turn inequality constraints in the optimization problem to equality constraints. As an example, consider the following constraint:

$$\overline{P_i} - \overline{L_i} > 0 \tag{61}$$

We suppose $\overline{P_i} - \overline{L_i} = \phi$ Therefore we can rewrite the corresponding constraint as:

$$\begin{gathered}\overline{P_i} - \overline{L_i} = \emptyset \\ \phi > 0\end{gathered} \tag{62}$$

As you can see, the number of K_i controllers is equal to the number of GA's chromosomes and we have one controller in each region, so if the number of regions is to be increased, the number of chromosomes will correspondingly increase and it won't interfere with the solution process. Solving optimization problem 2 can be done with similar manner that used for optimization problem 1.
We use two examples in order to establish the effectiveness of the proposed method in this paper. The parameter of GA for these examples are setting the percentage of crossover 65 and the percentage of mutation 15. In addition, the population size is 1000. In these examples, the cost function for optimization problem 1 is $J_!$ and another one is J_2

VI. NUMURICAL EXAMPLE

Example 1 Consider System (6) with grade 2 and $i = 1,2,3$ and the following matrices:

$$A_1 = \begin{pmatrix}0 & 1\\ 0 & -0.1\end{pmatrix}, A_2 = \begin{pmatrix}0 & 1\\ 1 & -0.1\end{pmatrix}, A_3 = \begin{pmatrix}0 & 1\\ 0 & -0.1\end{pmatrix} \tag{63}$$

$$a_1 = -a_3 = \begin{pmatrix}0\\ -1\end{pmatrix}, a_2 = \begin{pmatrix}0\\ 0\end{pmatrix}, B_1 = B_2 = B_3 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{64}$$

$$X_1 = \{x|x_1\epsilon[-2,-1]\}, X_2 = \{x|x_1\epsilon[-1,1]\}, X_3 = \{x|x_1\epsilon[1,2]\} \tag{65}$$

Suppose that the initial state $x_1(0)$ is a random variable with uniform distribution in the interval$[-2,2]$. We assume the cost function as equation (30) and assume $R_i = 1$ and$Q_i = 1$. We consider the control coefficient in interval$[-5,5]$.

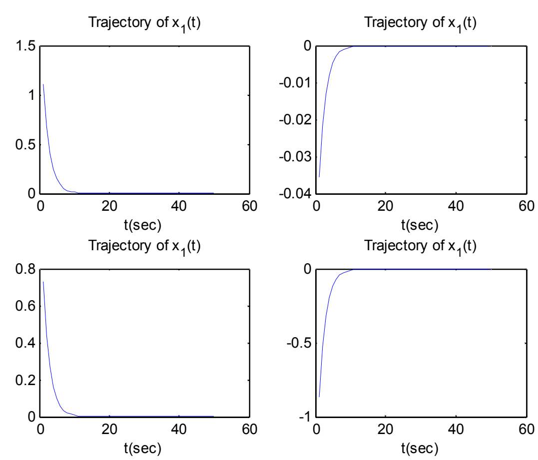

Fig.1. Trajectory of $\mathbf{X_1}$ for various initial conditions

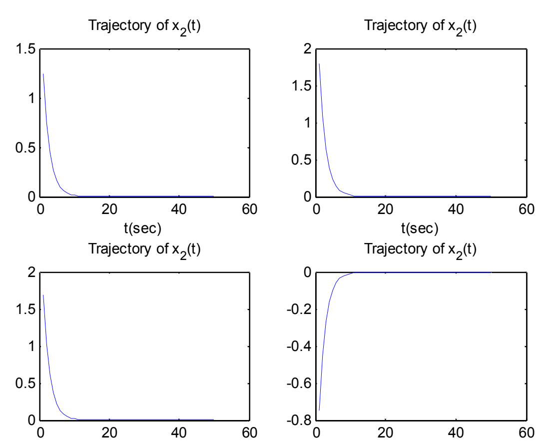

Fig.2. Trajectory of $\mathbf{X_2}$ for various initial conditions

It becomes clear that the origin is located in region X_2 and the closed loop system is unstable. Matrices $E_1 and\ E_2\ and\ E_3\ and\ e_1\ and\ e_2\ and\ e_3$ are calculated as:

$$E_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, e_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, e_3 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{66}$$

Also, the parameters required to analyze the stability using theorem (1) are:

$$C_{12} = (1 \quad 0), C_{23} = (1 \quad 0), F_{12} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, F_{23} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad c_{12} = (1); c_{23} = (-1), f_{12} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, f_{23} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{67}$$

We have done simulation for optimization problem 1 and optimization problem 2. After the simulation, the appropriate control coefficients are obtained as follows. Also, the appropriate Lyapunov function for region X_2 is demonstrated in figure (1):

$$K_1 = (\,0.6882 \quad 3.102) \tag{68}$$

$$K_2 = (-4.8810 \quad -3.3435) \tag{69}$$

$$K_3 = (-3.3782 \quad -3.3782) \tag{70}$$

$J_1 = 0.1698$

$J_2 = 1.9875$

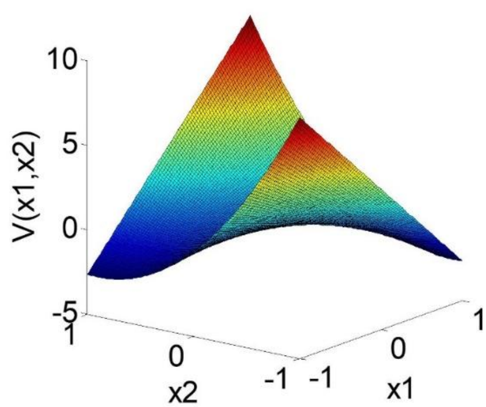

Fig.3. Lyapunov function for region X_2

By comparing J_1 and J_2 it is obvious that the optimization problem 1 is more efficient, this means that discontinuous quadratic lyapunov functions is better that continuous quadratic lyapunov function.

Example 2 Consider System (6) with grade 2 and $i = 1,2,3$ and the following matrices:

$$A_1 = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & 0.8 \\ -0.7 & -2 \end{pmatrix}, A_3 = \begin{pmatrix} -1 & 0.8 \\ -0.7 & -2 \end{pmatrix} \tag{71}$$

$$a_1 = -a_3 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, a_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, B_1 = B_2 = B_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{72}$$

$$X_1 = \{x|x_1\epsilon[-2,-1]\}, X_2 = \{x|x_1\epsilon[-1,1]\}, X_3 = \{x|x_1\epsilon[1,2]\} \tag{73}$$

Suppose that the initial state $x_1(0)$ is a random variable with uniform distribution in the interval $[-2,2]$. We assume the cost function as equation (30) and assume $R_i = 1$ and $Q_i = 1$. We consider the control coefficient in interval$[-5,5]$.

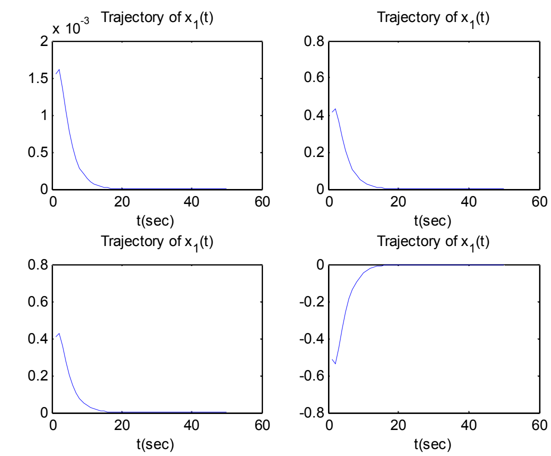

Fig.4. Trajectory of $\mathbf{X_1}$ for various initial conditions

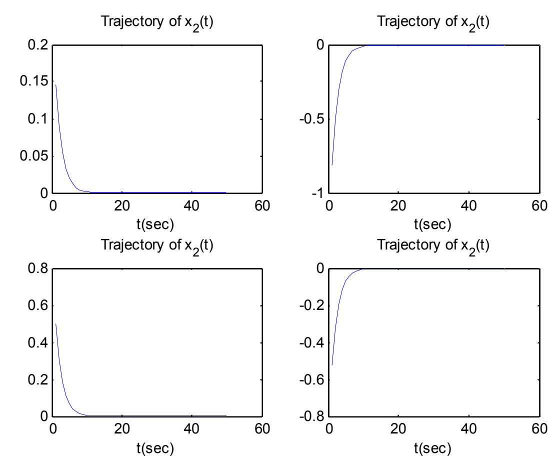

Fig.5. Trajectory of $\mathbf{X_2}$ for various initial conditions

It becomes clear that the origin is located in region X_2 and the closed loop system is unstable. Matrices $E_1\ and\ E_2\ and\ E_3\ and\ e_1\ and\ e_2\ and\ e_3$ are calculated as:

$$E_1 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad e_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, e_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, e_3 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{74}$$

Also, the parameters required to analyze the stability using theorem (1) are:

$$C_{12} = (1 \quad 0), C_{23} = (1 \quad 0), F_{12} = \binom{0}{1}, F_{23} = \binom{0}{1}$$
$$c_{12} = (1); c_{23} = (-1), f_{12} = \binom{-1}{0}, f_{23} = \binom{1}{0} \quad (75)$$

After the simulation, the appropriate control coefficients are obtained as follows. Also, the appropriate Lyapunov function for region X_2 is demonstrated in figure (4):

$$K_1 = (0.6882 \quad -2.2397) \quad (76)$$
$$K_2 = (1.5510 \quad -3.1029) \quad (77)$$
$$K_3 = (-3.5239 \quad -3.8100)$$
$$J_1 = 0.0094 \quad (78)$$
$$J_2 = 1.1873$$

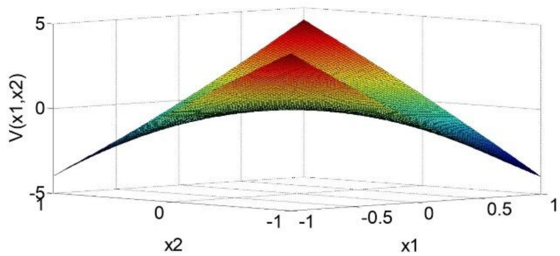

Fig.6. Lyapunov function for region X_2

VII. CONCLUSIONS

In this paper, a class of hybrid systems that are able to model a wide range of practical systems is introduced and after providing the mathematical description and stability conditions of PWA systems in the form of LMIs, the upper bound of the cost function is calculated. In fact, theorem4, theorem 5 and theorem 6 are the innovations of this article which prove that the problem of optimal control of PWA systems leads to BMI problem. Then, by minimizing the upper bound and using of GA and semi-definite programming, the controller coefficients are obtained. Note that we don't use the GA for solving the optimization problem, in fact GA use for searching the acceptable solution. The importance of what is done lies in the fact that semi-definite programming is used to solve the optimization problem 1 and optimization problem 2 and this has less error than other methods. Considering the proposed method for optimal control is a comprehensive method, one can apply this method design optimal control for practical systems. In addition, in these two optimization problems because of the assumption of the continuity of the piecewise linear system, it lacks sliding mode which is a benefit of these design and makes these methods very suitable for designing optimal controllers for electronic power converters. In the end, we use two numerical examples to establish the effectiveness of the discussed methods. In two examples, simulation results show that optimization problem 1 that is based discontinuous quadratic lyapunov function is more efficient that one.

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