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A numerical scheme for constrained optimal control problems

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ABSTRACT

In this paper, a numerical technique is proposed to solve optimal control problems (OPCs) of Volterra integral equations (VIEs). We apply the linear B-spline polynomials to solve OPCs by VIEs. The B-spline function divides the interval into sub-intervals and then built a different approximating polynomial on each sub-interval. In this method, optimal trajectory and control functions are expanded in terms of B-spline functions. The linear B-spline operational matrix of integration and multiplication are utilized in the proposed method. The main characteristic this method is that by using the suggested numerical technique and the related operational matrices, optimal control problem governed by Volterra integral equations is converted to a system of equations. Suffice it to say that this scheme simplifies the main problems and also makes to obtain a good approximate solution for them. In the end, there are two illustrative examples which numerical results show the validity and applicability of our method.

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I. INTRODUCTION

Optimal control theory is an important area of applied mathematics that was introduced by Pontryagin and collaborators in the 1950s.Optimal control theory has already found applications in many areas of science and engineering such as biologymedicine, economics, and finance. There are two main classes of optimal control problems (OCPs) that can be governed bydifferential equations or integral equations. Analytical and numerical methods exist for solving the various OCPs.

Therefore, recently, OCPs have been attracted attention of many researchers to obtain solutions to these problems. For more study one can see [1]-[6].There are several methods for solving differential equations[7]-[11].Among all of the techniques, orthogonal functions and polynomials have been extensively utilized for solving OCPs.Because these polynomials and functions have high accuracy. Numerous method based on orthogonal polynomials has attempted to derive solutions of OCPsFor example,Elnagar and Razzaghi [12] in 1997s studied a pseudo spectral Legendre method for linear quadratic optimal control problems. In 2011, Maleknejad et al. [13] applied triangular functions for solving OCPs governed by VIEs.A similar argument has been obtained by Tohidi&Samadi [14].Hat functions (HFs) is used to solve linear and non-linear integral equations [15]-[16].In [17] optimal control problems including integro-differential equations is solved using Hybrid functions. In [18], Hermite wavelet is applied to solve optimal control problem. In the current paper, we suggest the linear B-spline polynomials to solve optimal control problems governed by integral equations. B-spline function is divided the interval into a collection of subintervals and construct a different approximating polynomial on each subinterval. Main applications of B-splines appear in geometric modeling, computer aided design, computer graphics and many other different subjects [19].In this scheme, the state variables and the control variables are approximated with a linear B-spline

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functions. To this end, we use Operational matrix of integration. In consequence, optimal control problems convert to systems of algebraic equations. With the result, the state variables and the control variables is obtained.

This paper is organized as follows: Section 2 contains a brief summary of linear B-spline functions on [0,1] and approximation of function. Also the operational matrix of fractional integration is computed. In Sections 3, the proposed method is used to approximate OCPs. Section 4 describes the proposed method for solving some examples. Finally, we conclude with a summary in last Section.

II. B-Spline Function and Operational Matrix of Integration

A. *Linear B-spline function on [0,1]*

The m th-order cardinal B-spline $N_m(t)$ has the knot sequence $\{\cdots,-1,0,1,\cdots\}$. Also there are polynomials of order m (degree $m-1$) between the knots. The B-spline functions for $m \geq 2$on $[0,1]$has the following form [19-23]:

$$N_m(x) = \frac{1}{(m-1)!}\sum_{k=0}^{m}\binom{m}{k}(-1)^k(x-k)_+^{m-1}. \quad (1)$$

Where $\text{supp}[N_m(x)] = [0,m]$and the characteristic function for$m=1$ is $N_1(x) = \chi_{[0,1]}(x)$. The explicit presentation of $N_2(x)$ (the linear B-spline function) is defined by De Boor et al. [19-21] as following:

$$N_2(x) = \sum_{k=0}^{2}\binom{2}{k}(-1)^k(x-k)_+$$

$$= \begin{cases} x, & x \in [0,1), \\ 2-x, & x \in [1,2), \\ 0, & elsewhere. \end{cases} \quad (2)$$

Where

$$(x-k)_+ = \begin{cases} x-k, & x \geq k, \\ 0, & x < k. \end{cases} \quad (3)$$

Assume that $N_{j,k}(x) = N_2(2^jx-k), j,k \in \mathbb{Z}$ and $B_{j,k}(x) = Supp[N_{j,k}] = close\{x: N_{j,k} \neq 0\}$. It is obvious that their support is:

$$B_{j,k}(x) = [2^{-j}k, 2^{-j}(2+k)], \qquad j,k \in \mathbb{Z}. \quad (4)$$

In the light of these functions use on $[0,1]$,we define

$$S_j = \{k: B_{j,k} \cap [0,1] \neq 0\}, \qquad j \in \mathbb{Z}.$$

It is clear that $\min\{S_j\} = -1$and $\max\{S_j\} = 2^j-1, j \in \mathbb{Z}$. Since we need these functions on$[0,1]$, therefore we consider:

$$\phi_{j,k}(x) = N_{j,k}(x)\chi_{[0,1]}(x), \quad j \in \mathbb{Z}. \quad (5)$$

Accordingly, the linear B-spline scaling functions for $k = 0,1,\cdots,2^j-2$ can be written by

$$\phi_{j,k}(x) = \sum_{i=0}^{2}\binom{2}{i}(-1)^i\left(2^jx-(k+i)\right)_+$$

$$= \begin{cases} 2^jx-k, & \frac{k}{2^j} \leq x < \frac{k+1}{2^j}, \\ 2-(2^jx-k), & \frac{k+1}{2^j} \leq x < \frac{k+2}{2^j}, \\ 0, & elsewhere, \end{cases} \quad (6)$$

The respective left and right boundary scaling functions for $k = -1, 2^j-1$are:

$$\phi_{j,-1}(x) = \begin{cases} 1-2^jx, & 0 \leq x < \frac{1}{2^j}, \\ 0, & o.w, \end{cases} \quad (7)$$

and

$$\phi_{j,2^j-1}(x) = \begin{cases} 2^jx-2^j+1, & 1-\frac{1}{2^j} \leq x < 1, \\ 0, & o.w. \end{cases} \quad (8)$$

B. *The function approximation*

For a fixed $j = J$, The expansion of $f(x) \in [0,1]$ with respect to linear B-spline functions can be approximated as:

$$f(x) = \sum_{k=-1}^{2^J-1} c_k\phi_{J,k}(x) = C^T\Phi_J(x). \quad (9)$$

Where$C$and $\Phi_J(x)$are (2^J+1)vectors as:

$$C = [c_{-1}, c_0, \cdots, c_{2^J-1}]^T, \quad (10)$$

$$\Phi_J(x) = [\phi_{-1}, \phi_0, \cdots, \phi_{2^J-1}]^T. \quad (11)$$

Then

$$C^T = \left(\int_0^1 f(x)\Phi_J^T(x)dx\right)P^{-1}, \quad (12)$$

And symmetric matrix is given as:

$$P = \int_0^1 \Phi_J(x)\Phi_J^T(x)dx$$

$$= \frac{1}{2^{J-2}}\begin{bmatrix} \frac{1}{12} & \frac{1}{24} & 0 & & \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{24} & \frac{1}{6} & \frac{1}{24} \\ & & & \frac{1}{24} & \frac{1}{12} \end{bmatrix}. \quad (13)$$

C. *Operational matrix of integration*

In this subsection, the operational matrix of integration is obtained. Integral of vector $\Phi_J(x)$can be derived as:

$$\int_0^x \Phi_J(x)dx = I_\phi\Phi_J(x). \quad (14)$$

Where I_ϕ is $(2^J+1)\times(2^J+1)$ operational matrix of Integral for the linear B-spline function on$[0,1]$ that can be obtained as follows:

$$I_\phi = \left(\int_0^1\left(\int_0^x \Phi_J(x)dx\right)\Phi_J^T(x)dx\right)P^{-1} = EP^{-1}. \quad (15)$$

Where

$$E = \int_0^1\left(\int_0^x \Phi_J(x)dx\right)\Phi_J^T(x)dx. \quad (16)$$

By using Equ. (14) and Equ. (16)

$$E = \frac{1}{2^{2J+2}} = \begin{bmatrix} \frac{1}{4} & \frac{11}{12} & 1 & \cdots & \cdots & 1 & \frac{1}{2} \\ \frac{1}{12} & 1 & \frac{23}{12} & 2 & \cdots & 2 & 1 \\ & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & \ddots & \ddots & 2 & \vdots \\ & & & \ddots & 1 & \frac{23}{11} & 1 \\ & & & & \frac{1}{12} & 1 & \frac{11}{12} \\ & & & & & \frac{1}{12} & \frac{1}{4} \end{bmatrix}_{(2^J+1)(2^J+1)} \quad (17)$$

Therefore, I_ϕ is operational matrix of integration based on B-spline function as:

$$I_\phi = EP^{-1}. \tag{18}$$

D. Product perational matrix

The product operational matrix $\hat{C}$ of the linear B-spline function is given by

$$C^T\Phi_J(x)\Phi_J^T(x) = \Phi_J^T(x)\hat{C}. \tag{19}$$

Where $\hat{C}$ is $(2^J+1)(2^J+1)$ matrix. For more information about operational matrix of product, refer to [24].

III. The Propose Method

This section is focused on the following class of optimal control problems by Volterra integral equations (VIEs):

$$J(v,u) = \int_0^1 \Psi\big(x, v(x), u(x)\big)\,dx, \tag{20}$$

subject to

$$v(x) = w(x) + \int_0^x \kappa\big(x, y, v(y), u(y)\big)dy, \tag{21}$$

where $x \in [0,1]$, and $\Psi\big(x, v(x), u(x)\big) = u^2(x) + v^2(x) + f(x)v(x) + g(x)u(x)$ where $f(x), g(x)$ are real functions in $L^2[0,1]$. Optimal control problem is determining the optimal control and the corresponding optimal state satisfying (21) while minimizing the cost function (20). To solve the problem (20)-(21), it is assumed that $v(x), u(x), w(x)$ and $\kappa\big(x, y, v(y), u(y)\big)$ are:

$$v(x) = V^T\Phi_J(x),$$
$$u(x) = U^T\Phi_J(x),$$
$$w(x) = W^T\Phi_J(x), \tag{22}$$
$$\kappa\big(x, y, v(y), u(y)\big) = \Phi_J^T(x)K\Phi_J(x).$$

Where $\Phi_J(x)$ was defined by (11) and

$$V = [v_{-1}, v_0, \cdots, v_{2^J-1}]^T,$$
$$U = [u_{-1}, u_0, \cdots, u_{2^J-1}]^T,$$
$$W = [w_{-1}, w_0, \cdots, w_{2^J-1}]^T,$$

By substituting above equations in (21), we have

$$V^T\Phi_J(x) = W^T\Phi_J(x) + \Phi_J^T(x)K\int_0^x \Phi_J(y)dy, \tag{23}$$

By using operational matrix of integration and operational matrix of product, we have

$$V^T\Phi_J(x) = W^T\Phi_J(x) + \Phi_J^T(x)KI_\phi\Phi_J(x) = W^T\Phi_J(x) + \widehat{H}^T\Phi_J(x). \tag{24}$$

Where $A = K.I_\phi$ and $\widehat{H}$ is operational matrix of product. Consequently, (24) can be expressed in term of vectors V and U which has been called $\Phi^*(V,U)$.

Now, we approximate functions $f(x)$ and $g(x)$ in Equ. (20) as:

$$f(x) = F^T\Phi_J(x), g(x) = G^T\Phi_J(x), \tag{25}$$

F and G are the linear B-spline function coefficients of $f(x)$ and $g(x)$. Substituting (22), (25) in (20), we have got:

$$J(V,U) = \int_0^1 \big(V^T\Phi_J(x)\Phi_J^T(x)V + U^T\Phi_J(x)\Phi_J^T(x)U + F^T\Phi_J(x)\Phi_J^T(x)V + G^T\Phi_J(x)\Phi_J^T(x)U\big)\,dx. \tag{26}$$

By using (13):

$$J(V,U) = V^TPV + U^TPU + F^TPV + G^TPU. \tag{27}$$

Let

$$J^*(V,U) = J(V,U) + \Phi^*(V,U)\lambda. \tag{28}$$

Where $\lambda = [\lambda_{-1}, \cdots, \lambda_{2^J-1}]$ is the unknown Lagrange multiplier. The following conditions for the minimum are given by

$$\frac{\partial J^*}{\partial V} = 0, \quad \frac{\partial J^*}{\partial U} = 0, \quad \frac{\partial J^*}{\partial \lambda} = 0.$$

By solving this system, the approximate values of $v(x)$ and $u(x)$ from Equ. (21) will be obtained.

IV. Applications

Example1: Consider the following non-linear optimal control problem

$$J(v,u) = \int_0^1 \big((v(x) - x - 1)^2 + (u(x) - x^2 - x)^2\big)\,dx, \tag{29}$$

subject to

$$v(x) = w(x) + \int_0^x yx^2u(x)v(x)dy, \tag{30}$$

where

$$w(x) = -\frac{1}{5}x^7 - \frac{1}{2}x^6 - \frac{1}{3}x^5 + x + 1. \tag{31}$$

The exact optimal trajectory and control functions are in the following form:

$$v(x) = x + 1, \qquad u(x) = x^2 + x,$$

and minimum value J is $J^* = 0$.

This problem is solved by the suggested technique with different values of J. The obtained approximate solutions for both the state variable $v(x)$ and the control variable $u(x)$ together with the exact solutions are presented for $J = 3, 5$ in Figures 1 and 2 respectively. Absolute Error of $v(x)$ and $u(x)$ for different values of $J = 3, 5$ are shown in Tables 1 and 2. The optimal values of J^* for various values of J using the suggested method are listed in Table 3.

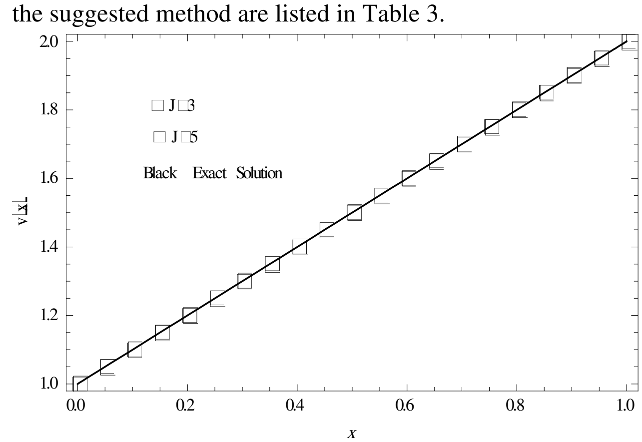

Fig. 1.Exact solution and obtained approximate solution of $v(x)$ for $J = 3,5$.

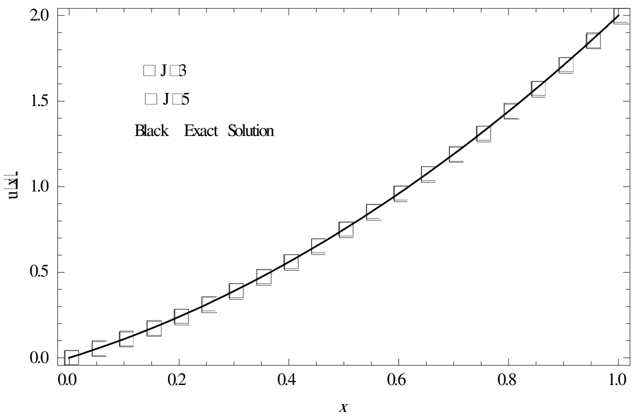

Fig. 2.Exact solution and obtained approximate solution of $u(x)$for $J = 3,5$.

Table1. Absolute Error of $v(x)$ for $J = 3,5$ in Example 1

x	J=3	J=5
0.0	7.78602×10^{-7}	6.15457×10^{-10}
0.1	1.0498×10^{-6}	3.84529×10^{-9}
0.2	2.30611×10^{-6}	1.0132×10^{-8}
0.3	5.04679×10^{-6}	1.96066×10^{-8}
0.4	8.37936×10^{-6}	3.27138×10^{-8}
0.5	7.96199×10^{-6}	5.01976×10^{-8}
0.6	2.93962×10^{-5}	7.34133×10^{-8}
0.7	3.40489×10^{-6}	1.04666×10^{-7}
0.8	6.85843×10^{-5}	1.47224×10^{-7}
0.9	9.02191×10^{-5}	3.23741×10^{-7}
1	3.39718×10^{-4}	5.54449×10^{-6}

Table 2. Absolute Error of$u(x)$ for $J = 3,5$ in Example 1

x	J=3	J=5
0.0	2.60418×10^{-3}	1.6276×10^{-4}
0.1	1.0389×10^{-4}	6.50937×10^{-6}
0.2	1.14641×10^{-3}	7.16169×10^{-5}
0.3	1.14681×10^{-3}	7.16183×10^{-5}
0.4	1.02751×10^{-4}	6.50486×10^{-6}
0.5	2.60247×10^{-3}	1.62752×10^{-4}
0.6	1.00373×10^{-4}	6.4985×10^{-6}
0.7	1.14903×10^{-3}	7.16334×10^{-5}
0.8	1.15535×10^{-3}	7.1646×10^{-5}
0.9	8.52475×10^{-5}	6.45093×10^{-6}
1	2.59003×10^{-3}	1.62707×10^{-4}

Table 3.The optimal values of J^*at different values of J for Example 1.

J	J^*
3	1.36165×10^{-6}
4	8.481×10^{-8}
5	5.29848×10^{-9}

Example2: Consider the following non-linear optimal control problem

$$J(v,u) = \int_0^1 ((v(x) - e^x)^2 + (u(x) - e^x)^2)\, dx, \quad (32)$$

subject to

$$v(x) = w(x) + \int_0^x yx^2 u(x) v(x) dy, \quad (33)$$

where

$$w(x) = e^x \left(1 - x - \tfrac{1}{2} e^x\right) + x + \tfrac{1}{2}. \quad (34)$$

For this example the exact optimal trajectory and control functions are in the following form:

$v(x) = u(x) = e^x$,

and minimum value J is $J^* = 0$.

The proposed technique is applied for OCPs by VIEs with different values of J. The obtained results for the state variable $v(x)$ and the control variable $u(x)$ with the exact solutions for $J = 4, 6$ are presented in Figures 3 and 4, respectively. Absolute Error of $v(x)$ and $u(x)$ for different values of $J = 4,6$ are shown in Tables 4 and 5. The optimal values of J^* for various values of J using the suggested method are listedin Table 6. It is apparent from the figures and tables that by increasing the value of J of B-spline basis, the approximate values of $v(x)$ and $u(x)$ will converge to the exact solutions.

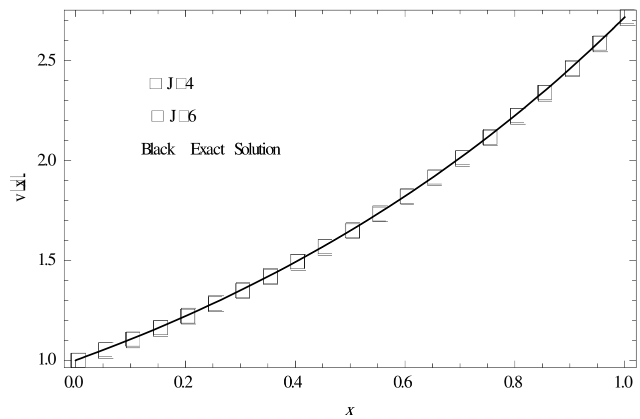

Fig. 3.Exact solution and obtained approximate solution of $v(x)$for $J = 4,6$.

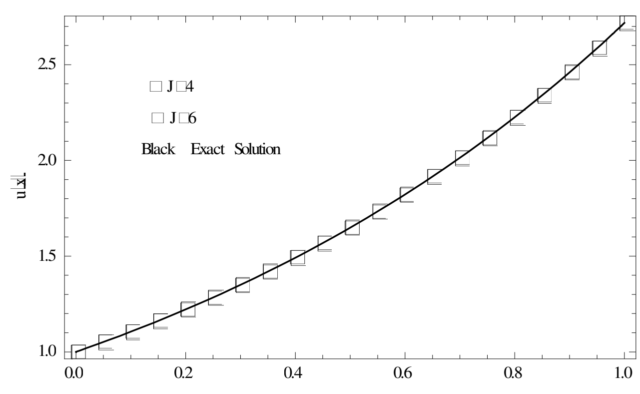

Fig. 4.Exact solution and obtained approximate solution of $u(x)$for $J = 4,6$.

Table 4. Absolute Error of $v(x)$ for $J = 4,6$ in Example 2

x	$J = 4$	$J = 6$
0.0	3.32372×10^{-3}	2.04545×10^{-5}
0.1	1.56374×10^{-4}	9.92626×10^{-6}
0.2	1.12194×10^{-4}	1.06918×10^{-6}
0.3	2.3081×10^{-5}	1.01705×10^{-6}
0.4	2.16165×10^{-4}	1.33074×10^{-5}
0.5	5.37078×10^{-4}	3.35449×10^{-5}
0.6	2.56637×10^{-4}	1.63637×10^{-5}
0.7	1.93949×10^{-5}	1.76575×10^{-6}
0.8	4.01416×10^{-5}	1.68154×10^{-6}
0.9	3.51347×10^{-4}	2.19326×10^{-5}
1	8.42000×10^{-4}	5.46163×10^{-5}

Table 5. Absolute Error of $u(x)$ for $J = 4,6$ in Example 2

x	$J = 4$	$J = 6$
0.0	3.30167×10^{-4}	2.04183×10^{-5}
0.1	1.56331×10^{-4}	9.92661×10^{-6}
0.2	1.11231×10^{-4}	1.06875×10^{-6}
0.3	2.29356×10^{-5}	1.01648×10^{-6}
0.4	2.16356×10^{-4}	1.33081×10^{-5}
0.5	5.36823×10^{-4}	3.35438×10^{-5}
0.6	2.56988×10^{-4}	1.63652×10^{-5}
0.7	1.88678×10^{-5}	1.76371×10^{-6}
0.8	3.87509×10^{-5}	1.67846×10^{-6}
0.9	3.54736×10^{-4}	2.19373×10^{-5}
1	8.73123×10^{-4}	5.51078×10^{-5}

Table 6.The optimal values of J^*at different values of J for Example 2.

J	J^*
3	2.17284×10^{-6}
4	1.35489×10^{-7}
6	5.28932×10^{-10}

V. CONCLUSIONS

This paper set out with the aim of assessing the solutions of optimal control problems by Volterra integral quations. To this end we utilized the linear B-spline polynomials. Also using the properties of operational matrices of B-spline, the cost of computational is low. The results corroborate that the proposed technique is very simple and effective.

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