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# Dynamic Modelling and Optimal Control of a Rotor of Active Magnetic Bearings

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ABSTRACT

*Active magnetic bearings (AMBs) are relatively new members in bearings family. Against other bearings which support loads by forces produced by fluid film pressure or physical contact, AMBs support loads by magnetic fields without any contact with the shaft and make it to be levitated. Because of that feature, AMBs have lots of benefits in comparison with other bearings. This paper presents modelling of AMBs with one and two degrees of freedom and the necessary parameters and equations for control of them is driven. Then, a numerical method based on orthogonal functions called Direct Method for optimal control in AMBs is presented with or without inequality constraints. The approach consists of reducing the optimal control problem with a two-boundary-value differential equation to a set of algebraic equations by approximating the state and control variables. This approximation uses orthogonal functions with unknown coefficients. In addition, the inequality constraints are converted to equal constraints. The problems are solved using Legendre and Haar bases. Simulation results demonstrate the benefits of Legendre functions method in compared with those using Haar bases.*

**Article Info**

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## I. INTRODUCTION

Generally magnetic bearings operate contactlessly and are therefore free of lubricant and wear. New technologies in the fields of energy efficiency, contactless operating condition, no sealing constraints and frictionless suspension are facilitating development of more sustainable with high speed applications. They are suited for applications with high rotation speeds such as compressors, turbines, pumps, motors and generators. The forces operate through an air gap, which allows magnetic suspension through hermetic encapsulations [1]. A major classification of magnetic bearings can be made into active and passive bearings. The main difference between active and passive bearings is that passive magnetic bearing (PMBs) have permanent magnets and do not need electric power to operate. On the other hand, they present low power loss because of the lack of current and active control ability and low damping stiffness [2], while active magnetic bearings (AMBs) use electromagnets that require an electric power input. They have better control ability and high stiffness, whereas they suffer from high power loss due to the biased current [3]. Then, active bearings are electrically controlled with some type of controller, while passive bearings are not electrically powered and thus have no control system. Therefore, AMBs are more efficient, fast and have a prolonged life. The advent of active magnetic bearings (AMBs) demonstrates the commitment of bearing industry to promoting technologies such as no contact between bearings, and consequently, no need for lubricants.

Smart-bias design to the complementary flux mode of operation [4], linear quadratic regulator (LQR) [5, 6], Linear Quadratic Gaussian (LQG) [7-9] have been considered for optimal control in AMBs. Chen et al. [10] used fuzzy gain tuning strategy dealing with the problem of unbalanced vibration problem to control an AMB system. In addition, genetic algorithms were used for auto-tuning of PID controllers to control AMBs [11-13]. Since most industrial AMBs are controlled by manual-tuning PID controllers, the presented approach produced an optimal design to tune the PID parameters. The PID controller is however only effective at the vicinity of the rotor's equilibrium position in linear dynamics of the AMB system and therefore, significant deviation of the rotor's motion from this equilibrium position

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may occur due to large imbalance forces [14].

In order to control the rotor position on the axial direction of a thrust active magnetic bearing, a Hermite polynomial based recurrent neural network strategy was developed by Lin et al [15]. The proposed method was able to control the rotor position for the tracking of various reference trajectories on the axial direction. In addition, a recurrent wavelet fuzzy neural network with adaptive differential evolution was presented to minimize the energy consumed by an AMB without altering its positioning performance and robustness [16]. The proposed method demonstrated the accuracy control and energy saving performances.

Nonlinear modelling [17] was used to analyse and simulate an AMB. Simulations of AMB backup bearing systems were significantly carried out the ability to correctly predict the occurrence of forward whirl for a well-balanced and axially levitated rotor.

Complex decoupling control problems have been analysed in a new hybrid permanent magnet type bearingless motor [18]. It has integrated the competencies of hybrid excitation permanent magnet motor and magnetic bearings. The method has advantages of both the simple control mechanism and the low power consumption.

Mushi et al. [19] presented a design and modelling framework for flexible rotors supported by AMBs. A μ-synthesis controller was developed for the rotor-AMB model. The levitation response and output sensitivity measurements, illustrated improved stability and performance over the PID law. Also, a large-order, flexible shaft rotor dynamics model for a flywheel supported with magnetic bearings was modelled to develop a magnetic suspension compensator to provide satisfied reliability and disturbance rejection [20]. Finally, closed loop mathematical model with structural variables and controller states was constructed and simulated and therefore, it demonstrated the properties of robustness to the standard sensor run out patterns, mass imbalances and inertial load impact.

Lee and Jeong [21] developed a fully coupled linearized dynamic model for a cone-shaped AMB system including the relationships between input voltage and output current in the cone shaped magnet coil. The results showed the efficiency of analytical developments in the stabilization and tracking performance of the controlled system. An equivalent magnetic circuit model was built to analyse the design parameters for a novel heteropolar radial hybrid magnetic bearing [22]. The optimizations on both structure and key design parameters were proposed to reduce its control current and consequently to decrease of alternating frequency of bias magnetic flux. Mathematical modelling [23] was used to analyse and simulate a sequential quadratic programming method. The presented model was implemented for the optimization design of a radial hybrid magnetic bearing. A dynamic model of a rotor [24] was presented to suspend the high speed rotor stably. Also, a decoupling control method was introduced to decouple the radial translations mode and tilting mode of rotor. Then, in order to suppress the rotor's vibration, a displacement cross feedback controller was used.

A discrete-time constrained model predictive control (MPC) algorithm was used to control and stabilize a flexible rotor supported on AMBs subject to input and output constraints [25]. The approach was developed to niche MPC scheme to stabilize the MIMO rotor AMB system by regulating the output displacement to zero.

Nonlinear controllers [26, 27] were developed in experimental applications of AMBs. Also, an adaptive backstepping neural controller was used to achieve precise rotor position tracking of an AMB system with modelling uncertainties and external disturbances [28]. To approximate the unknown nonlinearities of dynamic systems, a single hidden layer feedforward network was developed. Sliding mode controllers were designed to control the AMB systems operating such that the systems can achieve robust performance under various system nonlinearities [29, 30]. In addition, adaptive sliding mode controllers [31-33] were used to control the position of the rotor of AMBs. The controllers guarantee both stability and performance robustness and verify the effectiveness of the proposed method for static and dynamic mass imbalances. Notch filter techniques [34-37] were considered on the feedback control loop to reduce the synchronous forces transmitted through the bearings.

The cost function may consist of rotor position, electric energy, kinetic energy, or a combination of them. The main goal of the optimization of power management would be to minimize the cost function. Therefore, an optimization algorithm must be used. An approach to solve this problem is based on optimal theory, calculus of variations, and Pontryagin maximum principle. Casas [38] used the Pontryagin maximum principle and introduced a penalty problem by using the Ekeland principle to solve an optimal control problem with state constraints that were verified by semi linear parabolic equations. To use this method, a two boundary value problem needs to be solved. However, solving this would be very difficult through an analytic approach.

Orthogonal functions have received considerable attention for dealing with various problems related to dynamic systems. The main benefit of using orthogonal functions in optimal control is that the differential equations are reduced to a set of algebraic equations, and thus there is no need to solve two boundary value problems. Therefore, the optimal solution is directly extracted by solving algebraic equations. These approximation algorithms are known as "Direct Methods" [39]. Depending upon the structure of the orthogonal functions, they are classified into three groups:

a. Piecewise constant orthogonal functions such as Haar wavelets and block-pulse functions.

b. Orthogonal polynomials such as Legendre, Chebyshev, and Jacobi.

c. Sine-Cosine functions such as Fourier transform.

In this paper, the direct method is used to solve the optimal control of a modelled AMB system with and without inequality constraints. This method consists of reducing the optimal control problem with a two-boundary-value differential equation to a set of algebraic equations by approximating the state and control variables which are the beginning and end positions of the rotor, their velocities, and the currents. This approximation uses two groups of orthogonal functions such as Legendre polynomials and Haar wavelet functions with unknown coefficients. The operational matrix of product and the operational matrix of integration are used to evaluate the unknown coefficients. Also, the inequality constraints are transferred into algebraic equalities [40]. Therefore, the necessary conditions for the optimal power management are derived as algebraic equations in the unknown coefficients of the state and control variables and Lagrange multipliers. These coefficients are determined in such a way that the necessary conditions for extremization are imposed.

The paper is organised as follows. Section 2 presents the basic formulation and modelling of the optimal control problem for single and double sided AMBs with (without) inequality constraints. Section 3 describes approximation of the optimal control problem for AMBs through the direct method using Haar and Legendre functions. Section 4 presents simulation results for the considered models. Finally, conclusions are given in Section 5.

## II. Problem Statement

### A. Single sided AMB model

Demands for higher velocities in bearing without friction led to advent of AMBs developments (Fig. 1). Nowadays, the most widely spread concept for contactless levitation are AMBs [41].

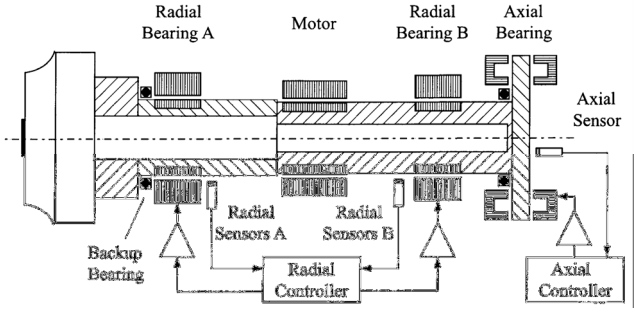



Fig. 1. An AMB scheme with axial and radial veering and current control [41]

A two pole electromagnetic actuator of an AMB is shown in Fig. 2 [42]. From the electrical and mechanical model of the AMB, a 1 DoF[1] of the system is presented as follows:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ \frac{k_x}{m} & 0 & \frac{k_i}{m}\\ 0 & -\frac{k_i}{L} & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0\\ 0\\ \frac{1}{L}\end{bmatrix}u \tag{1}$$

where

$$[x_1\ x_2\ x_3]^T=[y\ \ \dot{y}\ \ i]^T$$

$$u=v \tag{2}$$

and $y$, $\dot{y}$, $i$, $v$, $m$, $L$, $k_x$ and $k_i$ are displacement of rotor, speed of rotor, control current, voltage input in coil, rotor's total mass, force displacement factor of bearing, corresponding force current factor and self-inductance of the coils [42].

### B. Double sided AMB model

For each 2 DoFs in AMBs, there are two opposing differential driving mode electromagnets. A model description of a 2 DoF AMB is shown in Fig. 3. In order to model the AMB, a simple scheme of Fig. 3 is shown in Fig. 4 where $y$, $\theta$, $y_1$, $y_2$, $F_1$ and $F_2$ are: displacement of rotor's center of gravity G, rotation angle of the rotor, displacement of beginning and the end of the rotor and forces respectively [42].

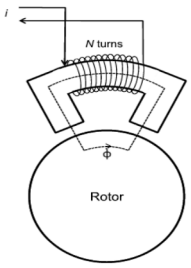



Fig. 2. A two pole electromagnetic actuator [42]

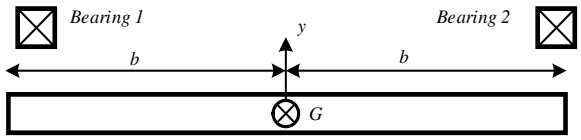



Fig. 3. Schematic view of an AMB system (radial dynamics)

As can be seen from the figure and considering mechanical and electrical dynamics in the AMB, the dynamic model of the system is presented as follows [43]:

$$\ddot{y}_1=k_x\left(\frac{1}{m}+\frac{b^2}{J}\right)y_1+k_i\left(\frac{1}{m}+\frac{b^2}{J}\right)i_1+k_x\left(\frac{1}{m}-\frac{b^2}{J}\right)y_2+k_i\left(\frac{1}{m}-\frac{b^2}{J}\right)i_2 \tag{3}$$

$$\ddot{y}_2=k_x\left(\frac{1}{m}-\frac{b^2}{J}\right)y_1+k_i\left(\frac{1}{m}-\frac{b^2}{J}\right)i_1+k_x\left(\frac{1}{m}+\frac{b^2}{J}\right)y_2+k_i\left(\frac{1}{m}+\frac{b^2}{J}\right)i_2$$

[1] Degree of Freedom

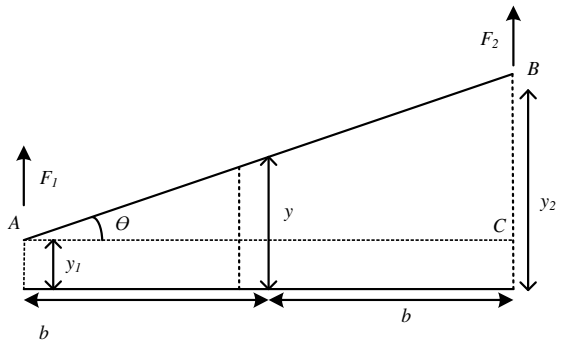



Fig. 4. Simple scheme of a 2 DoF rotor in AMB

By substituting

$$M_1 = \frac{1}{m} + \frac{b^2}{J}$$
$$M_2 = \frac{1}{m} - \frac{b^2}{J} \quad (4)$$

and

$$[x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ u_1\ u_2]^T = [y_1\ y_2\ \dot{y}_1\ \dot{y}_2\ i_1\ i_2\ v_1\ v_2]^T \quad (5)$$

the state space model of the AMB is achieved as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ k_x M_1 & k_x M_2 & 0 & 0 & k_i M_1 & k_i M_2 \\ k_x M_2 & k_x M_1 & 0 & 0 & k_i M_2 & k_i M_1 \\ 0 & 0 & -\frac{k_i}{L} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{k_i}{L} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{L} & 0 \\ 0 & \frac{1}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (6)$$

where $\dot{y}_1$ and $\dot{y}_2$ are derivation of $y_1$ and $y_2$. $i_1$ and $i_2$ represent control current that are to be applied to the coils and $v_1$ and $v_2$ are voltage inputs in coils. $m$, $J$, $k_x$, $k_i$ and $L$ are rotor's total mass, its radial moment of inertia, force displacement factor of the bearing, corresponding force current factor and self-inductance of the coils.

In order to control of the rotor position in AMB, an optimal control strategy can be employed to minimize the energy used by the coils. This paper presents an optimal method of power management for the AMBs to reduce their electrical energy consumption while the control of the rotor's position is considered. It is assumed that the optimal problem includes a cost function, state equations, initial states, and some equality or inequality constraints. These time variant parameters can be linear or nonlinear, and also discrete or continuous. The problem is solved using the direct method approach.

To have an optimal solution for the power management in AMBs, it is necessary to introduce a cost function. This cost function should be minimized to achieve an optimal solution. There are some known cost functions that could be considered including minimum time, minimum control effort, tracking, or a combination of these [44]. In this paper, due to minimize the electrical energy and position control, the cost function is formulated as:

$$J_a = \int_0^{t_f} (X^T Q X + U^T R U)\, dt \quad (7)$$

where $X$, and $U$ are state and control (input) vectors and $Q = diagonal(q_1,\ldots,q_6)$ and $R = diagonal(r_1, r_2)$ represent weight matrices.

Therefore, the problem is to find an admissible control variable $u_1(t)$ and $u_2(t)$ subject to state equation (Equ. (6)) that minimizes the cost function. It may concern that both rotor's position and control inputs should vary in their allowed boundary limitations. The problem becomes how to find $u_1(t)$ and $u_2(t)$ that minimize the cost function of Equ. (7) subject to the state equation of Equ. (6) and the inequality constraints.

## III. Approximation of the System Using Orthogonal Functions

In this paper the direct method is used to find the minimum electrical energy consumption needed in the AMBs in order to control the rotor's position. One of the main aspects of this method is that it reduces the optimal power management problem in AMBs to those of solving a system containing algebraic equations, and thus the problem can be solved greatly simpler. This method is based on converting the differential equations into algebraic equations by approximating all parameters involved in the problem by truncated hybrid functions and also, the integral operations can be eliminated by using the operational matrix of integration [39].

In addition, although various computing techniques have been used to solve the unconstrained optimal control problems, they have both analytical and computational difficulties while solving the optimal control problems with inequality constraints [45]. However, in the presented approach, the inequality constraints are transferred into a set of algebraic equations. Then, using undetermined Lagrange multiplier, the constraints are adjoined to the final cost function.

Suppose

$$B = [b_0(t)\ b_1(t)\ \cdots\ b_{n-1}(t)]^T \tag{8}$$

is the base vector of order $n$ direct method, where $n$ is the number of orthogonal functions such as Legendre polynomials or Haar wavelet. These bases are introduced in $[0, t_f]$ as follow [40]:

$$b_n(t) = \begin{cases} P_m(\frac{2N}{t_f}t - 2n + 1), & t \in [\frac{n-1}{N}t_f, \frac{n}{N}t_f] \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

where

$$P_0(t) = P_1(t) = 1, \tag{10}$$
$$P_{m+1}(t) = \left(\frac{2m+1}{m+1}\right)tP_m(t) - \left(\frac{m}{m+1}\right)P_{m-1}(t)$$

is a recursive formula for orthogonal functions. A function such as $f(t)$ that is defined in the interior $[0, t_f]$, can be approximated as follows:

$$f(t) \cong \sum_{i=1}^{n-1} c_i b_i(t) = C^T B \tag{11}$$

where

$$C = [c_0, ..., c_{n-1}]^T, \tag{12}$$

is the coefficient vector. Also, the integration of $B$ can be approximated by

$$\int_0^t B d\tau \approx PB \tag{13}$$

where $P$ is a $n \times n$ operational matrix for integration [39]. $P$ can be obtained from an approximation of the bases integration of order $(n+1)$ to order $n$. The product of two hybrid function vectors can be approximated using hybrid functions as follows:

$$BB^T C \approx \tilde{C}B \tag{14}$$

where $\tilde{C}$ is a $n \times n$ product operational matrix.

Since the cost function of Equ. (7) should be minimized by an acceptable $u_1(t)$ and $u_2(t)$ ($u(t)$ for the double sided AMB model), the following estimations are applied in the system using Equ. (11):

$$\begin{cases} \dot{x}_i(t) = B^T X_i, & i = 1, \ldots, 6 \\ u_i(t) = B^T U_i, & i = 1, 2 \end{cases} \tag{15}$$

where $X_i$ and $U_i$ are unknown vectors. It may be noted that the method is presented for the double sided AMB model and can consequently consider for the single sided AMB model.

Also, with the same rule:

$$x_i(0) = B^T X_{0i}, \ i = 1, \ldots, 6 \tag{16}$$

where $X_{0i}$ is a known vector of order $n$. Also, $x_i(t)$ is obtained by integrating $\dot{x}_i$ from 0 to $t_f$ as follows:

$$x_i(t) - x_i(0) = \int_0^{t_f} B^T X_i \cong B^T P X_i, \ i = 1, \ldots, 6. \tag{17}$$

Then,

$$x_i(t) = B^T(PX_i + X_{0i}), \ \ i = 0, \ldots, 6 \cdot \tag{18}$$

By substituting the approximated vectors in the cost function, Equ. (7) is converted to:

$$\begin{aligned} J_a = \int_0^{t_f} ([B^T(PX_1 + X_{01})]^T q_1 [B^T(PX_1 + X_{01})] + \cdots + \\ + [B^T(PX_6 + X_{06})]^T q_6 [B^T(PX_6 + X_{06})] + \\ + [B^T U_1]^T r_1 [B^T U_1] + [B^T U_2]^T r_2 [B^T U_2]) dt \end{aligned} \tag{19}$$

In order to solve the problem using hybrid functions, the integration of the cost function should be transferred into an algebraic equation, and time should be extracted from the cost function [39].

In order to simplify the problem, the isoperimetric constraint can be devised as a differential equation constraint by differentiating it with respect to time. Therefore, Equ. (6) can be rewritten using the direct method as follows

$$\begin{bmatrix} B^T X_1 \\ B^T X_2 \\ B^T X_3 \\ B^T X_4 \\ B^T X_5 \\ B^T X_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ k_x M_1 & k_x M_2 & 0 & 0 & k_i M_1 & k_i M_2 \\ k_x M_2 & k_x M_1 & 0 & 0 & k_i M_2 & k_i M_1 \\ 0 & 0 & -\frac{k_i}{L} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{k_i}{L} & 0 & 0 \end{bmatrix} \begin{bmatrix} B^T(PX_1 + X_{01}) \\ B^T(PX_2 + X_{02}) \\ B^T(PX_3 + X_{03}) \\ B^T(PX_4 + X_{04}) \\ B^T(PX_5 + X_{05}) \\ B^T(PX_6 + X_{06}) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{L} & 0 \\ 0 & \frac{1}{L} \end{bmatrix} \begin{bmatrix} B^T U_1 \\ B^T U_2 \end{bmatrix} \tag{20}$$

therefore:

$$\hat{X}=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ k_x M_1 & k_x M_2 & 0 & 0 & k_i M_1 & k_i M_2 \\ k_x M_2 & k_x M_1 & 0 & 0 & k_i M_2 & k_i M_1 \\ 0 & 0 & -\frac{k_i}{L} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{k_i}{L} & 0 & 0 \end{bmatrix}\begin{bmatrix} (PX_1+X_{01}) \\ (PX_2+X_{02}) \\ (PX_3+X_{03}) \\ (PX_4+X_{04}) \\ (PX_5+X_{05}) \\ (PX_6+X_{06}) \end{bmatrix}+\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \frac{1}{L} & 0 \\ 0 & \frac{1}{L} \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \quad (21)$$

Also by introducing auxiliary functions [46], the inequality constraints are transferred into the equality constraints such as follows:

$$\left|x_1(t)\right| \le x_{1_{\max}} \xrightarrow{yields} x_1(t)-x_{1_{\max}}+z_1^2(t)=0 \quad (22)$$

where $z_1(t)$ is auxiliary function. By expanding $z_1(t)$ in term of the direct method, it is given as:

$$z_1(t)=B^T Z_1. \quad (23)$$

Then,

$$z_1^2(t)=Z_1^T BB^T Z_1=Z_1^T \tilde{Z}_1 B \quad (24)$$

where $\tilde{Z}_1$ is calculated from the product operational matrix of Equ. (14). Also, $x_{1_{\max}}$ is approximated by the following hybrid functions:

$$x_{1_{\max}}=B^T X_{1_{\max}} \quad (25)$$

where $X_{1_{\max}}$ is a known vector. Therefore, Equ. (21) is converted to

$$C_1^T B=\left(X_1^T-X_{1_{\max}}^T+Z_1^T\tilde{Z}_1\right)B=0. \quad (26)$$

To add this constraint to the basic minimization problem, it is converted to another minimization problem in which $x_1(t)$ should be find to minimize:

$$J_C=\int_0^{t_f}\{C_1^T Bw_1 B^T C_1\}dt \quad (27)$$

where $w_1>0$ is a weight function that can be scalar or matrix. Here, for simplicity it is assumed as scalar constant. Since $C_1>0$ is not related to time and $w_1>0$ is constant, $J_C$ is transformed to:

$$J_C=w_1 C_1^T L C_1 \quad (28)$$

where

$$L=\int_0^{t_f} BB^T dt \cdot \quad (29)$$

The minimization problem is replaced by the following parameter optimization problem [39]:

Find the coefficients of $\lambda$, $Z_i$, $X_i$, $i=1,\ldots,6$ and $U_i$, $i=1,2$ which minimize the cost function:

$$J_{tot}=J_a+J_C+\lambda^T\left((I-A)\hat{X}-B\hat{U}\right) \quad (30)$$

where $\lambda$ is an unknown Lagrange multiplier vector. The determining algebraic equations for these vectors are:

$$\frac{\partial J_{tot}}{\partial Z_1}=0, \quad (31)$$

$$\frac{\partial J_{tot}}{\partial X_i}=0,\ i=1,\ldots,6, \quad (32)$$

$$\frac{\partial J_{tot}}{\partial U_i}=0,\ i=1,2, \quad (33)$$

and

$$\frac{\partial J_{tot}}{\partial \lambda}=0. \quad (34)$$

## IV. Simulation Results

### *A. Optimal control of a single sided AMB model without inequality constraint*

To evaluate the effect of the proposed method, two types of orthogonal functions, Legendre polynomials and Haar bases, were carried out. Legendre polynomials are orthogonal on the interval [-1, 1], [47]. To use these bases within [0, 1], a transferred form of Legendre polynomials and Haar wavelet functions should be used. Thus:

$$B_0(t)=1, B_1(t)=2t-1,\ldots,$$
$$,B_n(t)=\left(\frac{2n-1}{n}\right)(2t-1)B_{n-1}(t)-\left(\frac{n-1}{n}\right)B_{n-2}(t) \quad (35)$$

$$H(t)=\begin{cases} 1 & 0\le t<\frac{1}{2} \\ -1 & \frac{1}{2}\le t<1 \\ 0 & otherwise \end{cases}, \quad H_{jk}(t)=H(2^j t-k) \quad (36)$$

Design parameters of the single sided AMB model are summarized in Table I.

TABLE I

MAIN DESIGN PARAMETERS OF THE SINGLE SIDED AMB MODEL

| *Item* | *Type Size* |
|---|---|
| $m\ (kg)$ | 3 |
| $[x_{01}(0)\ x_{02}(0)\ x_{03}(0)]$ | [–0.005 0 0] |
| $k_x$ | 3924 |
| $k_i$ | 29.43 |
| $L\ (H)$ | 0.22 |
| $Q$ | $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| $R$ | 1 |

The problem is solved by the direct method and the results are shown in Fig. (5) and Fig. (6). Considering the results of the problem using Legendre polynomials and Haar bases on the, based on the calculated cost values (Table II), it is evident that the results using Legendre polynomials are more accurate than the results using Haar bases. In addition, other advantage of the proposed method is that its computational complexity is less than that of the Haar bases.

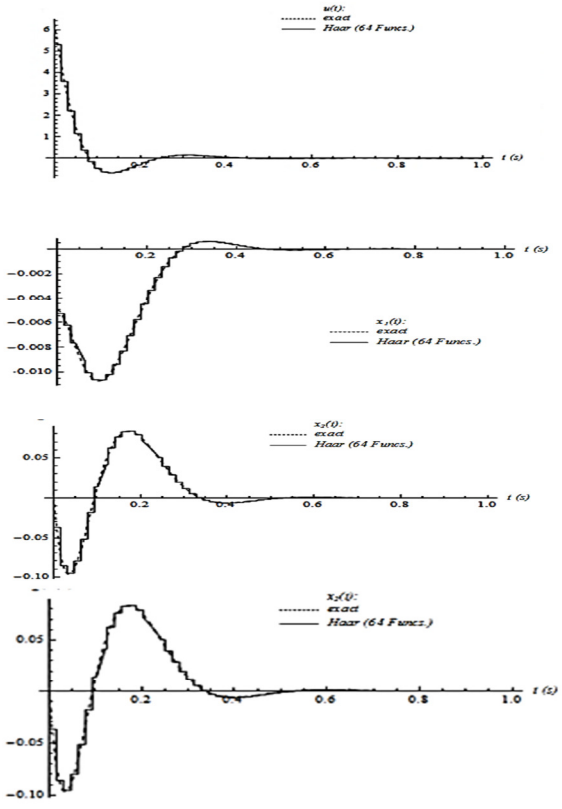



Fig. 5. State and control results using 64 bases of Haar bases

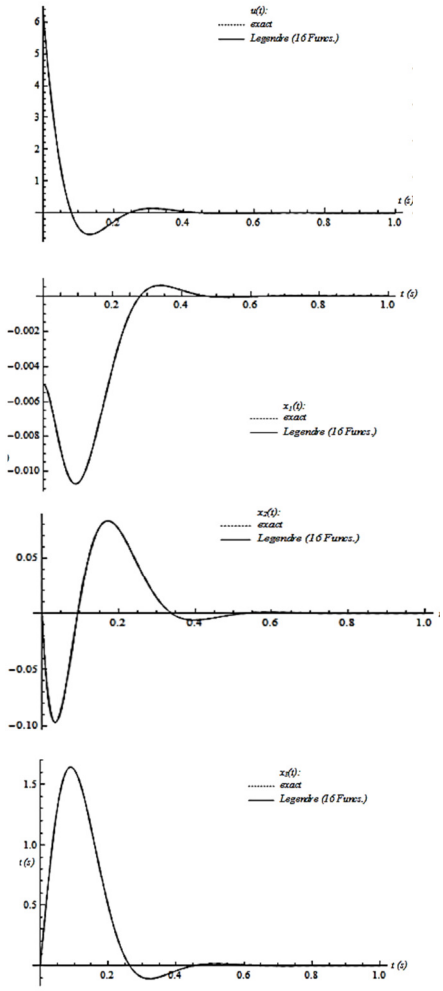



Fig. 6. State and control results using 16 bases of Legendre polynomials

TABLE II
CALCULATED COST VALUES

| *Method* | *Cost value* |
|---|---|
| *Exact solution* | *1.0653* |
| *Legendre Funcs.* | *1.0656* |
| *Haar Funcs.* | *1.0731* |

### *A. Optimal control of a single sided AMB model with inequality constraint*

Consider the optimal problem of previous section with inequality constraint $|x_1(t)| \le 0.0085$ . Similarly, the problem is solved by the direct method and the results for $x_1(t)$ are shown in Fig. (7) and Fig. (8).

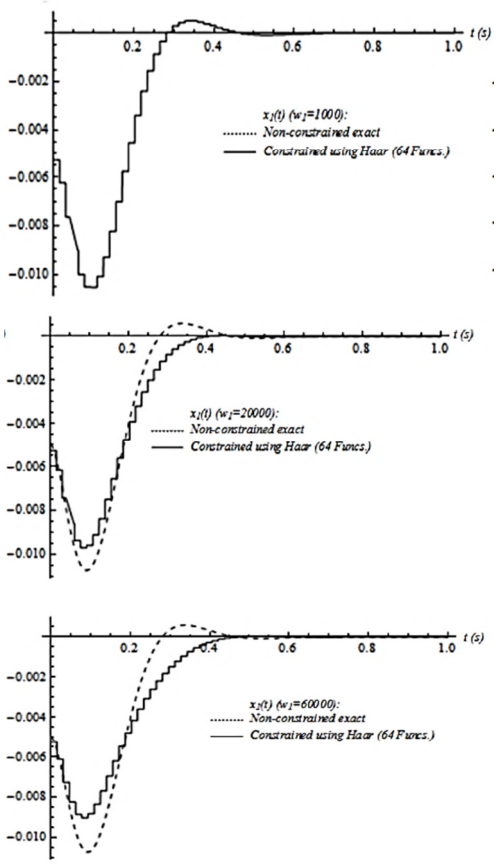



Fig. 7. Simulation results for $x_1(t)$ using 64 bases of Haar bases

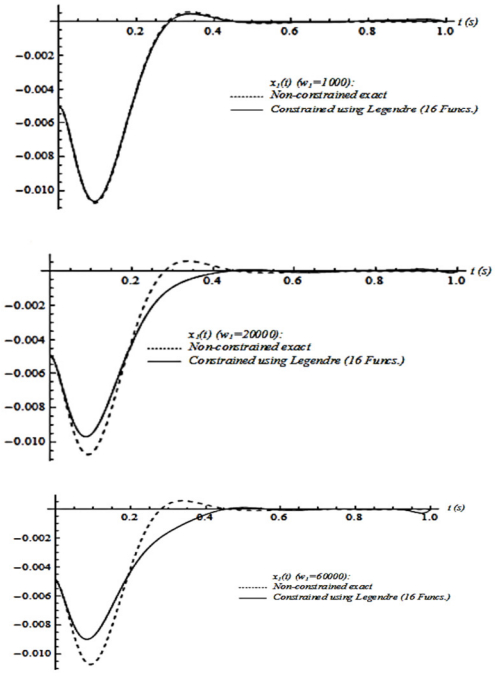



Fig. 8. Simulation results for $x_1(t)$ using 16 bases of Legendre polynomials

The same as previous section, the problem is solved by the direct method using Haar bases and Legendre polynomials. Considering the results, based on the calculated cost values (Table III), it is clear that the results using Legendre polynomials are more accurate than Haar bases. In addition, other advantage of the proposed method is that its computational complexity is less than that of the Haar bases. It may be noted that, since there is an inequality constraint for $x_1(t)$, the cost values increased and the amount of weight constant will affect the cost value.

TABLE III

CALCULATED COST VALUES

| *Method* | $w_1$ | *Cost value* |
|---|---|---|
| *Exact solution* | - | *1.0721* |
| *Legendre Funcs.* | *1000* | *1.0723* |
| | *20000* | *1.0945* |
| | *60000* | *1.135* |
| *Haar Funcs.* | *1000* | *1.0727* |
| | *20000* | *1.096* |
| | *60000* | *1.1389* |

### *B. Optimal control of a double sided AMB model without inequality constraint*

Design parameters of the double sided AMB model are summarized in Table IV. The problem is solved by the direct method and the results are shown in Fig. (9) and Fig. (10). Considering the results of the problem using Legendre polynomials and Haar bases on the, based on the calculated cost values (Table V), it is evident that the results using Legendre polynomials are more accurate than Haar bases. In addition, other advantage of the proposed method is that its computational complexity is less than that of the Haar bases (in this case, because of order of the dynamic model and computational complexity for Haar bases method, 32 functions are used).

TABLE IV

MAIN DESIGN PARAMETERS OF THE DOUBLE SIDED AMB MODEL

| *Item* | *Type Size* |
|---|---|
| $m\ (kg)$ | *6* |
| $[x_{01}(0)\cdots x_{06}(0)]$ | $[-0.006\ \ -0.004\ 0\ 0\ 0\ 0]$ |
| $k_x$ | *3924* |
| $k_i$ | *29.43* |
| $M_1$ | *0.5877* |
| $M_2$ | *-0.2544* |
| $Q$ | $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ |
| $R$ | *1* |

TABLE V

CALCULATED COST VALUES

| *Method* | *Cost value* |
|---|---|
| *Exact solution* | *2.23568* |
| *Legendre Funcs.* | *2.2393* |
| *Haar Funcs.* | *2.4778* |

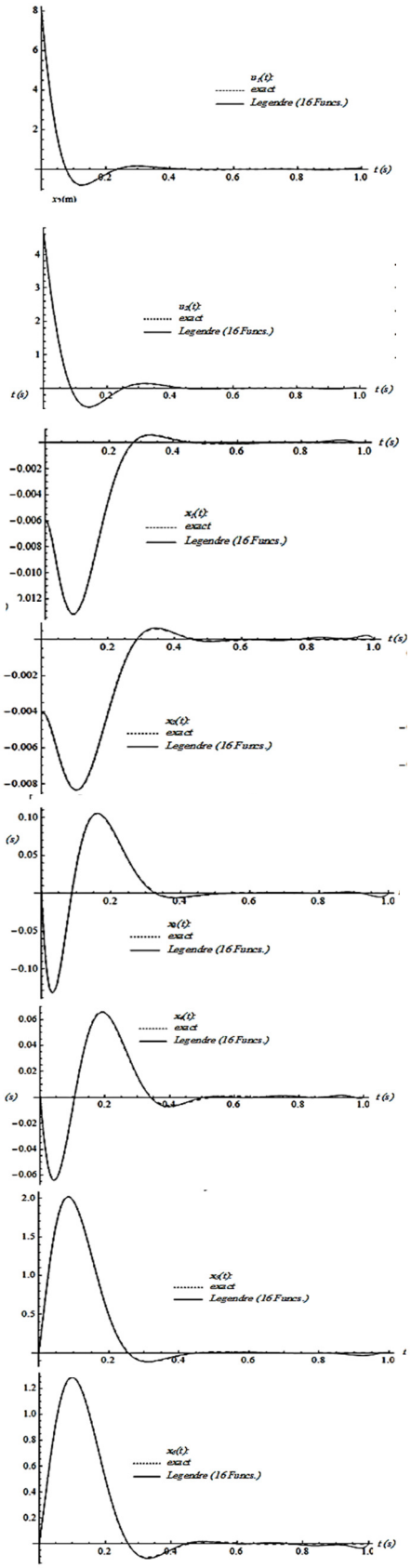



Fig. 9. State and control results using 16 bases of Legendre polynomials

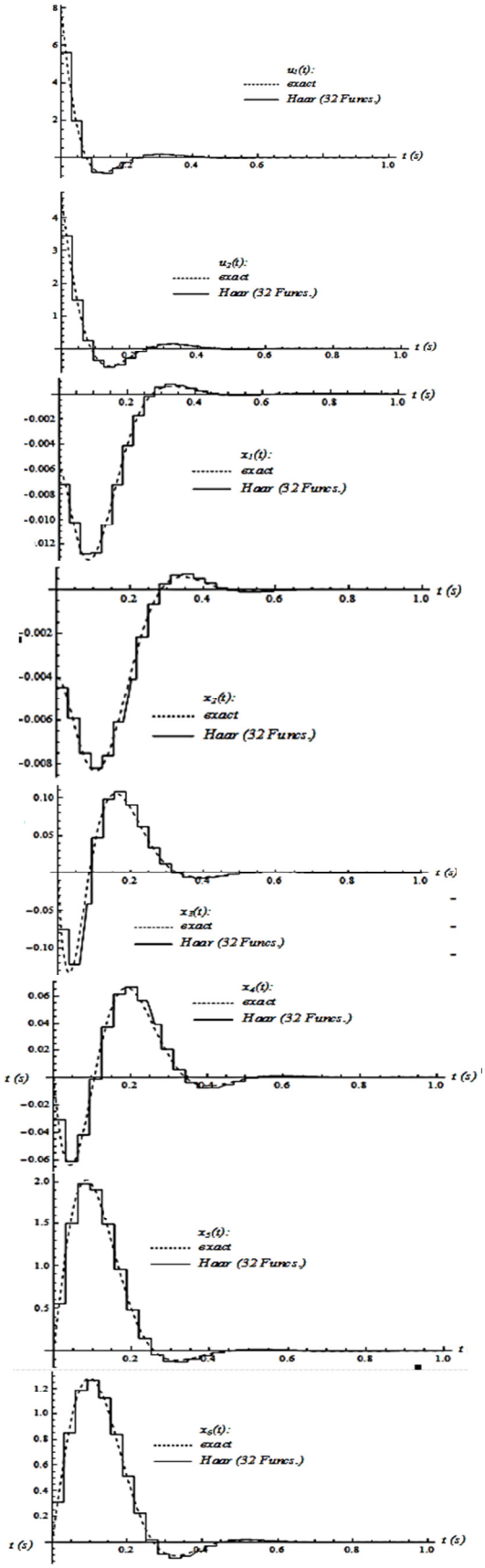



Fig. 10. State and control results using 32 bases of Haar bases

It is shown that the results of the hybrid functions method are close to the exact solution. Based on the cost values reported in the table, it is evident that the developed the direct method using Legendre functions is superior to the Haar bases approach.

## V. Conclusion

This paper presented a method for optimal control of active magnetic bearings to minimize the consumed power using hybrid functions. The method involves reducing the optimal control problem with a two-boundary-value differential equation to a set of algebraic equations by approximating the state and the control variables. Therefore, the necessary conditions for optimal control are derived as algebraic equations in the unknown coefficients of the state and control variables and Lagrange multipliers. These coefficients are determined in such a way that the necessary conditions for extremization are imposed. To demonstrate the efficiency of the developed method, two types (single and double sided) of AMBs were modelled and simulated with the direct method using Haar and Legendre bases. The results obtained by the proposed method were compared with the exact solution.

The results using the Legendre functions had much less difference than that of the Haar bases. In addition, the Legendre functions method gives better results because it is continuous. Therefore, more accurate results are obtained by the Legendre functions method.

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