A Novel Method for Optimal Control of Piecewise Affine Systems Using Semi-Definite Programming

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ABSTRACT

In this paper, a novel optimal control design method by discontinuous quadratic Lyapunov function and continuous quadratic Lyapunov function for 2-dimensional piecewise affine systems via semi-definite programming with LMI constraints is proposed. At the first, an upper bound for a quadratic cost function for a stable closed-system is obtained. Then after, considering a state-feedback control approach, not only sufficient conditions for the stability of the closed-loop system but also the upper bound of the cost function are obtained. The optimization problem is formulated as a semi-definite programming with bilinear constraints (BMI). Some variables in BMIs are searched by genetic algorithm, so the bilinear constraints are converted to linear constraints and the controller coefficients are calculated. The effectiveness of the proposed method is verified by numerical examples.

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I. INTRODUCTION

Hybrid systems are dynamical systems with continuous-time dynamics and discrete events. These systems are applied for modeling various real-world applications. Piecewise affine (PWA) systems are a subset of hybrid systems and their equivalence with some other classes of the hybrid systems are shown in [1]. As we know, many of nonlinearities such as saturation are either inherently modeled in the form of PWA systems or approximated as a PWA system [2]. Accordingly, the class of PWA systems is an important tool as well as a starting point for modeling and analysis of nonlinear systems. These systems are defined by partitioning the state-space into a finite number of polyhedral regions and associating a different affine dynamic model to each region. Power electronics, process control and a wide range of nonlinear systems in engineering are some of the attractive applications of PWA systems in recent years, for instance, see, [3-5] The excellence advantage of PWA systems is that the stability and performance analysis of such systems can be formulated as a convex problem which is easily solved by numerical methods. Controllability and observability of PWA systems discussed in [6]. In[7], stability analysis is expressed in the form of linear matrix inequalities (LMIs). In control theory, different approaches are proposed to define the optimal control law. Different algorithms for optimal control of continuous-time hybrid systems are compared in [8]. The necessary conditions for optimal control law in hybrid systems using dynamic programming and maximum principle can be found in [9]. To approximate optimal control law, [10] uses dynamic convex programming. Recently,
The dual of the above problem is defined as:

\[
\begin{align*}
\text{(D): } d^* &= \sup \left\{ b^T Y : \sum_{i=1}^{m} Y_i A_i + S = C, S \geq 0, Y \in R^m \right\} \\
\end{align*}
\]
Piecewise affine (PWA) systems

The mathematical description of PWA class system in general is [25]

\[ \begin{align*}
\dot{x} &= a_i + A_i x + B_i u \\
y &= c_i + C_i x + D_i u
\end{align*} \quad \text{for} \ x \epsilon X_i
\]

(6)

In which \( X_i \) are the corresponding regions and their collection is partitioned by the state space.

\[ X_i = \{ x \epsilon \mathbb{R}^2, E_i x \geq e_i \} \ \text{i} \epsilon 1 \]

(7)

In (7) \( E_i \) and \( e_i \) are respectively a matrix and a vector, with constant value and appropriate size.

It’s worth noting that if \( B_i = 0 \), the system (6) turns into a piecewise linear system as follows:

\[ \dot{x} = a_i + A_i x \quad \text{for} \ x \epsilon X_i \]

(8)

III. Stability Analysis

Stability of a system has a variety of definitions. In this paper, we call a system stable if it has global Lyapunov stability. The methods we use in order to evaluate the stability are discontinuous piecewise quadratic Lyapunov function [39] and continuous piecewise quadratic Lyapunov function [7].

For this, first we discuss the necessary background: Suppose the PWA system defined in the previous section, is continuous in its boundary, so for \( x \epsilon \overline{X}_i \cap \overline{X}_j \) we have: [39]

\[ A_i x + a_i = A_j x + a_j \]

(9)

If region \( X \) is partitioned as polyhedron in this case, each sub-region can be described as equation (10)

\[ \overline{X}_i = \{ x \epsilon \mathbb{R}^2 : E_i x \geq e_i \} \ \text{i} \epsilon 1 \]

(10)

Where \( E_i \) and \( e_i \) are respectively a matrix and a vector, with constant value and appropriate size. A parametric description of the boundary between two regions \( \overline{X}_i \) and \( \overline{X}_j \) where in \( \overline{X}_i \cap \overline{X}_j \neq \Phi \) can be described as

\[ \overline{X}_i \cap \overline{X}_j \subseteq \{ x \epsilon \mathbb{R}^2 : F_{ij} s + f_{ij} e \epsilon R \} \]

(11)

\[ \overline{X}_i \cap \overline{X}_j \subseteq \{ x \epsilon \mathbb{R}^2 : F_{ij} s_i + s e R \} \]

(12)

\[ E_i = [E_i - e_i] \]

(13)

\[ F_{ij} = \begin{pmatrix} f_{ij} & f_{ij} \\ 0 & 1 \end{pmatrix} \]

(14)

In this relation if \( F_{ij} \neq 0 \) the border is a part of a line and if \( F_{ij} = 0 \) the border is a point. To be more specific, you can refer [38].

For the two adjacent regions \( X_i \) and \( X_j \), we assume that \( F_{ij} \neq 0 \) (the border is part of a line) one can define vector \( \overline{C}_{ij} = [C_{ij} c_{ij}] \) and hyper plane line as \( s_{ij} = \{ x \epsilon R^2 : \overline{C}_{ij} \dot{x} = 0 \} \), which \( C_{ij} \) is the normal vector of \( s_{ij} \) (perpendicular to \( s_{ij} \)), with the direction from \( X_i \) to \( X_j \), so that \( \overline{X}_i \cap \overline{X}_j \subseteq s_{ij} \) is satisfied.

According to what was said, now we can propose the theorem regarding the stability of PWA systems. Consider the resulting Lyapunov function as the following relation:

\[ V(x) = V_1(x) = \dot{x}^T P_1 \dot{x} = \overline{P}_i e R_{3x3} \]

(15)

\[ r \epsilon R, q \epsilon R^2, P_i e R_{2x2} P_1 = \begin{pmatrix} p_i & q_i \\ q_i^T & r_i \end{pmatrix} \]

(16)

Theorem1: [39]

Suppose \( \overline{U}_i \) and \( \overline{W}_j \) are unknown matrices with non-negative elements and appropriate dimensions and \( k=1,2 \) \( \overline{W}_{ij}^{-k} \) are unknown vectors with appropriate dimensions and non-negative elements and (\( i \epsilon 1 \)) \( \overline{P}_i e R_{3x3} \) is a symmetric matrix, then define the following variables:

\[ \overline{H}_{ij} = E_i^T \omega_{\epsilon}^{-1} C_i \overline{A}_i + E_i^T \omega_{\epsilon}^{-2} C_i \overline{A}_i \]

\[ \overline{M}_i = E_i W_j E_i \]

(17)

If there is a choice between \( \overline{P}_i \) and \( \overline{U}_i \) and \( \overline{W}_i \) matrices and \( k=1,2 \) \( \overline{W}_{ij}^{-k} \) vectors that satisfy the following restrictions, then for system defined by equation (8) all the trajectory starting at \( X \) will exponentially converge to origin.

\[ \overline{P}_i - \overline{L}_i > 0 \quad \text{for} \ i \epsilon 1 \]

(18)

\[ (1_{n} 0)(\overline{P}_i - \overline{L}_i)(1_{n} 0) > 0 \quad \forall i \epsilon 0 \]

(19)

\[ \lambda_i^A \overline{P}_i + P_i A_i + M_i < 0 \quad \forall i \epsilon 1, i \epsilon 0 \]

(20)

\[ (1_{n} 0)(\lambda_i^A \overline{P}_i + P_i A_i + M_i)(1_{n} 0) < 0 \quad \forall i \epsilon 1 \]

(21)

\[ F_{ij}^T (P_i - \overline{P}_j) F_{ij} = F_{ij}^T (H_0 + H_j^T) F_{ij} \epsilon 1, j \in N_0, \text{where} F_j \neq 0 \]

(22)

\[ \text{Where} \ N_0 = \{ k \epsilon 1, k \neq i, \overline{X}_i \cap \overline{X}_j \neq \Phi \} \]

(24)

Theorem2: [7]

Consider symmetric matrices \( T_i \) and \( U_i \) and \( W_i \) such that \( U_i \) and \( W_i \) have nonnegative entries, while

\[ P_i = F_i^T T_i F_i, i \epsilon 1 \]

(25)

Satisfy

\[ \begin{cases} 0 > A_i^T P_i + P_i A_i + E_i^T U_i E_i, i \epsilon 0 \\
0 < P_i - E_i^T U_i E_i \end{cases} \]

(26)

\[ \begin{cases} 0 > A_i^T P_i + P_i A_i + E_i^T U_i E_i, i \epsilon 1 \\
0 < P_i - E_i^T U_i E_i \end{cases} \]

(27)

Then every continuous piecewise \( C^1 \) trajectory \( x(t) \) satisfying (8) for \( t > 0 \) tends to zero exponentially.
IV. CALCULATING UPPER BOUND

Theorem 3: [25]

For the system (8) if the conditions of theorem (1) are met and the inequality is established, then the upper bound for cost function \( j = \int_0^\infty x(t)^T Q x(t) dt \) is calculated:

\[
i \in I_0, P_k + A_i^T P_k + Q_i + E_i^T W_i E_i < 0 \tag{28}
\]

\[
i \in I_1, P_k + \overline{A_i}^T P_k + \overline{Q_i} + \overline{E_i}^T \overline{W_i} \overline{E_i} < 0 \tag{29}
\]

Proof: This proof is given from [25]. Suppose that \( i \in I_1 \) we prove theorem for \( i \in I_1 \), another proof is the same. By multiplying the said inequality by \( X \) from left and right and removing of non-negative terms. Then we take the integral of the expression that we desire in the interval \([0, \infty)\):

\[
\begin{align*}
\dot{x}^T P_k \dot{x} + \dot{x}^T \overline{Q} \dot{x} + \dot{x}^T \overline{E_i}^T \overline{W_i} \overline{E_i} \dot{x} &< 0 \\
\frac{d}{dt} (x^T P_k x + \dot{x}^T \overline{Q} \dot{x} + \dot{x}^T \overline{E_i}^T \overline{W_i} \overline{E_i} \dot{x}) &\geq 0 \\
\int_0^\infty \frac{d}{dt} (x^T P_k x + \dot{x}^T \overline{Q} \dot{x}) dt &\leq 0 \\
\frac{x^T P_k x}{0} + j &\leq 0 \\
0 - x(0)^T P_i x(0) + j &\leq 0 \\
j &\leq x(0)^T P_i x(0)
\end{align*}
\]

\[E(j) \leq E \left( \text{tr}(P_i x(0)^T x(0)^T) \right) = \sum_{i \in I} \alpha_i \text{tr}(P_i L_i) \tag{30}\]

\[L_i = \left\{ \begin{array}{l}
E(x_0 x_0^T) x_0 \in X_i, i \in I_0 \\
E(x_0 x_0^T) x_0 \in X_i, i \in I_1
\end{array} \right. \tag{31}\]

What is important for continuing this paper is (30) because we should use it for designing controller.

You may notice in the above inequalities that all of them are a series of LMI in relation to the variables \( P_k \) and \( (\overline{P}_k) \). Therefore, stability conditions for a closed-loop system are a series of LMI in relation to \( P_k \) and \( (\overline{P}_k) \) and they are convex optimization problems that can be solved using numerical methods. Note that because the cost function is dependent on the initial point and this point is an unknown random variable, we assume that it has a uniform distribution, so the dependency is eliminated. Operator \( E \) expresses the expected value and \( \alpha_i \) represents the probability that \( x_0 \) belongs to area \( X_i \). Since we considered the initial state as a uniform random variable, therefore the probability of \( \alpha_i \) and the covariance matrix \( L_i \) can be determined using the desired area’s information and the partition \( X_i \).

V. OPTIMAL CONTROL

In this section we describe the optimal controller design issues for PWA systems using the state feedback. We assume that the designated system balance point is the initial point. Consider the system described with equations (6), in this case assume that state feedback controller is \( u(t) = K_i x(t) \). The closed-loop system takes the form below:

\[
\begin{align*}
\ddot{x} &= (A_i + B_i K_i) x(t) + a_i \\
x(t_0) &= \overline{x}(t) + x(t_0) = \overline{x}(0)
\end{align*}
\]

We consider the cost function as:

\[
j(x_0, u) = \int_0^\infty [x(t)^T Q x(t) + u(t)^T R_i u(t)] dt
\]

With the consideration of the appropriate state feedback, the cost functions come in the form of:

\[
j(x_0, u) = \int_0^\infty [x(t)^T (Q_i + K_i^T R_i K_i) x(t)] dt
\]

\[
\begin{align*}
&j(x_0, u) \\
&= \int_0^\infty (x(t)^T 1) \left( Q_i + K_i^T R_i K_i \right) (x(t)) dt \\
&j(x_0, u) = \int_0^\infty x(t)^T Q_i x(t) dt
\end{align*}
\]

Using the notations of equation (35), the equation (29) gives:

\[
\begin{align*}
\dot{x}(t) &= \left( A_i + B_i K_i \right) x(t) \\
\overline{X}_i &= \left( A_i + B_i K_i \right) \overline{x}(t)
\end{align*}
\]

By applying the mentioned changes in the form of the equations, theorem 3 for the system (37) is rewritten as:

Theorem 4:

For the system (37) with (assuming that the system is stable) if the equations (39-40) are met, then the upper bound for the equation (36) is obtained:

\[
i \in I_0, P_k (A_i + B_i K_i) + (A_i + B_i K_i)^T P_k + Q_i + E_i^T W_i E_i < 0
\]

\[
i \in I_1, P_k + \overline{A_i}^T P_k + \overline{Q_i} + \overline{E_i}^T \overline{W_i} \overline{E_i} < 0
\]

In this case we’ll have: \( j \leq x(0)^T P_i x(0) \)

Proof: To prove this theorem in theorem 3, we convert \( A_i \) to \( A_i + B_i K_i \). Now we can merge theorems 1, theorem 2 and theorem 4 and generally express the result in terms of theorem 5 and theorem 6:

Theorem 5:

For the system defined by equations (37) if the following conditions are met, then the system for each respective system trajectory exponentially converges to the origin and
If $j \leq x(0)\,^TP_0x(0) \Rightarrow i \in I_0 \rightarrow \overline{L}_i = \frac{P_i}{0} > 0$  
(41)

\[
\overline{P}_i - \overline{L}_i > 0
\]

(42)

\[
\overline{P}_i - \overline{L}_i > 0
\]

(43)

\[
\overline{A}_i = \left( A_i + B_i \bar{K}_i \right)\frac{A_i}{0}
\]

(44)

\[
P_i(A_i + B_i K_i) + (A_i + B_i K_i)^T \overline{P}_i + Q_i + K_i^T R_i K_i + E_i^T W_i E_i < 0
\]

(45)

\[
i \in I_0, \overline{P}_i - \overline{L}_i > 0
\]

(46)

\[
\overline{A}_i - P_i A_i + \overline{L}_i < 0
\]

(47)

\[
P_i A_i + \overline{A}_i P_i + \overline{Q}_i + E_i^T W_i E_i < 0
\]

(48)

\[
A_i = \left( A_i + B_i \bar{K}_i \right)\frac{A_i}{0}
\]

(49)

\[
\forall i \in I_0, \overline{L}_i > 0
\]

(50)

\[
\overline{A}_i = A_i
\]

(51)

\[
(\overline{A}_i + B_i \bar{K}_i) + (A_i + B_i K_i)^T \overline{P}_i + Q_i + K_i^T R_i K_i + E_i^T W_i E_i < 0
\]

(52)

\[
\overline{F}_{ij} = F_{ij}
\]

(53)

\[
\overline{A}_i F_{ij} = A_i F_{ij}
\]

(54)

\[
\overline{E}(j) \leq E\left( \text{tr}(\overline{P}_i \overline{x}_i \overline{x}_i^T) \right) = \sum_{i \in I_0} a_i \text{tr}(\overline{P}_i \overline{L}_i)
\]

(55)

\[
L_i = \left\{ \begin{array}{ll}
E(x_0 \overline{x}_0^T) & x_0 \in X_i, i \in I_0 \\
E(\overline{x}_i \overline{x}_i^T) & x_0 \in X_i, i \in I_1
\end{array} \right.
\]

This theorem is based on discontinuous quadratic Lyapunov that is given in [39].

**Theorem 6:**
For the system defined by equations (37) if the following conditions are met, then the system for each respective system trajectory exponentially converges to the origin and $j \leq x(0)\,^TP_0x(0)$

\[
\begin{cases}
0 > A_i^T P_i + P_i A_i + E_i^T U_i E_i \ i \in I_0 \\
0 < P_i - E_i^T U_i E_i \\
P(A_i + B_i K_i) + (A_i + B_i K_i)^T \overline{P}_i + Q_i + K_i^T R_i K_i + E_i^T W_i E_i < 0
\end{cases}
\]

(56)

Finally, for optimal control design, the coefficient $K_i$ must be calculated. To calculate these coefficients we consider a controller that minimizes the upper bound of the cost function $\sum_{i \in I} a_i \text{tr}(\overline{P}_i \overline{L}_i)$. As we minimize the upper bound, the cost function will also be minimized; therefore, the desired optimization problem will actually lead to the design of the controller. In order to design optimal control we design optimal control based on theorem 5 and theorem 6. We give these two optimal control in the form of optimization problem 1 and optimization problem 2.

Optimization problem 1 (based on discontinuous quadratic Lyapunov function)

\[
\min \sum_{i \in I} a_i \text{tr}(\overline{P}_i \overline{L}_i)
\]

subject to

\[
\overline{K}_i \epsilon K
\]

Optimization problem 2 (based on continuous quadratic Lyapunov function)

\[
\min \sum_{i \in I} a_i \text{tr}(\overline{P}_i \overline{L}_i)
\]

subject to

\[
\overline{K}_i \epsilon K
\]

As you can see, these two optimization problems are actually a semi-definite programming problem with LMI and BMI constraints such as (48) and (56). In fact, they are two BMI problems, the references [32] use numerical V-K algorithms to solve problems with BMI constraints, but numerical algorithms V-K doesn’t have a good convergence and is trapped in local minimum. Note that our desired functions are not dependent on the variables $K_i$ and these variables can be seen in our objective function. Now if $K_i$ is given, then our optimization problems turn into a semi-definite programming problem. GA is a comprehensive solution for high dimensional problems. Suppose that the set $K$, is the set of all acceptable controllers for the controller coefficients $K_i$, if we find a way to calculate this coefficients, we have managed to design an optimal controller and the minimum value of cost function can be calculated. We have to calculate the controller coefficients using GA note that we don’t use GA for minimizing the cost function. In order to find these coefficients using the mentioned method, we ascribe each chromosome in GA to a corresponding controller coefficient $K_i$. In which case, the non-convex optimization problem turns into a semi-definite programming problem. Assuming that the controller coefficients $K_i$ are known, we can calculate the fitness function in each chromosome. If $K$ is ascribed to each chromosome, then the fitness function will be defined as:
In which $\bar{f}$ is the minimum cost function and can be easily calculated using CVX toolbox. Note that if $\bar{f}$ corresponds to an infeasible chromosome, then the minimum value of the cost function will be infinite and its fitness function will be zero, this will stop the corresponding chromosome with impossible answer from generating next generation of children. The algorithm routine used for two optimization problems are summarized as follows:

Step1: Initialize value of parameters related to the GA. (Specify the population size, percentage of crossover and percentage of mutation.)

Step2: Calculate the fitness function specified by the equation (59) for each chromosome or solution.

Step3: Using the calculated fitness function in step 2 and spinning the roulette wheel select one chromosome. Then, use the given values such as percentage of crossover and percentage of mutation to complete crossover and mutation.

Step4: If the algorithm termination conditions are met, extract the results, otherwise, go to step 1.

GA is an efficient algorithm for searching the best solution. You can follow this subject in the reference [40].

As you can see, the number of $K_i$ controllers is equal to the number of GA’s chromosomes and we have one controller in each region, so if the number of regions is to be increased, the number of chromosomes will correspondingly increase and it won’t interfere with the solution process. Solving optimization problem 2 can be done with similar manner that used for optimization problem 1.

We use two examples in order to establish the effectiveness of the proposed method in this paper. The parameter of GA for these examples are setting the percentage of crossover 65 and the percentage of mutation 15. In addition, the population size is 1000. In these examples, the cost function for optimization problem 1 is $J_1$ and another one is $J_2$

VI. NUMERICAL EXAMPLE

Example 1 Consider System (6) with grade 2 and $i = 1,2,3$ and the following matrices:

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & -0.1 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}$$

$$a_1 = -a_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B_1 = B_2 = B_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X_1 = \{x|x_1 \in [-2, -1]\}, X_2 = \{x|x_1 \in [-1, 1]\}, X_3 = \{x|x_1 \in [1, 2]\}$$

Suppose that the initial state $x_1(0)$ is a random variable with uniform distribution in the interval $[-2, 2]$. We assume the cost function as equation (30) and assume $R_i = 1$ and $Q_i = 1$.

We consider the control coefficient in interval $[-5,5]$.

![Fig.1. Trajectory of $X_1$ for various initial conditions](image1)

![Fig.2. Trajectory of $X_2$ for various initial conditions](image2)
the closed loop system is unstable. Matrices \( E_i \) and \( E_2 \) and \( E_3 \) and \( e_1 \) and \( e_2 \) and \( e_3 \) are calculated as:
\[
E_1 = \begin{pmatrix} 1 & \ 0 \\ -1 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & \ 0 \\ -1 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & \ 0 \\ -1 & 0 \end{pmatrix},
\]
\[
e_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, e_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, e_3 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.
\] (66)

Also, the parameters required to analyze the stability using theorem (1) are:
\[
C_{12} = (1 \ 0), C_{23} = (1 \ 0), F_{12} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, F_{23} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]
\[
c_{12} = (1), c_{23} = (-1), f_{12} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, f_{23} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\] (67)

We have done simulation for optimization problem 1 and optimization problem 2. After the simulation, the appropriate control coefficients are obtained as follows. Also, the appropriate Lyapunov function for region \( X_2 \) is demonstrated in figure (1):
\[
K_1 = (0.6882 \ 3.102)
\] (68)
\[
K_2 = (-4.8810 \ -3.3435)
\] (69)
\[
K_3 = (-3.3782 \ -3.3782)
\] (70)
\[
J_1 = 0.1698
\]
\[
J_2 = 1.9875
\]

By comparing \( J_1 \) and \( J_2 \) it is obvious that the optimization problem 1 is more efficient, this means that discontinuous quadratic lyapunov functions is better that continuous quadratic lyapunov function.

Example 2 Consider System (6) with grade 2 and \( i = 1,2,3 \) and the following matrices:
\[
A_1 = \begin{pmatrix} -1 & \ 0 \\ -1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & \ 0.8 \\ -1 & 0.8 \end{pmatrix}, A_3 = \begin{pmatrix} -1 & \ 0 \\ -0.7 & -2 \end{pmatrix}
\] (71)

\[a_1 = -a_3 = \begin{pmatrix} 0 & \ 0 \\ -1 & 0 \end{pmatrix}, a_2 = \begin{pmatrix} 0 & \ 0 \\ 1 & 0 \end{pmatrix}, B_1 = B_2 = B_3 = \begin{pmatrix} 1 & \ 0 \\ 0 & 1 \end{pmatrix}.
\] (72)

\[
X_1 = \{x|x_1 \epsilon [-2,-1]\}, X_2 = \{x|x_1 \epsilon [-1,1]\}, X_3 = \{x|x_1 \epsilon [1,2]\}
\] (73)

Suppose that the initial state \( x_1(0) \) is a random variable with uniform distribution in the interval \([-2,2]\). We assume the cost function as equation (30) and assume \( R_i = 1 \) and \( Q_i = 1 \). We consider the control coefficient in interval\([-5,5]\).

![Trajectory of \( X_1 \) for various initial conditions](image1)

![Trajectory of \( X_2 \) for various initial conditions](image2)

It becomes clear that the origin is located in region \( X_2 \) and the closed loop system is unstable. Matrices \( E_1 \) and \( E_2 \) and \( E_3 \) and \( e_1 \) and \( e_2 \) and \( e_3 \) are calculated as:
\[
E_1 = \begin{pmatrix} 1 & \ 0 \\ -1 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & \ 0 \\ -1 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 1 & \ 0 \\ -1 & 0 \end{pmatrix},
\]
\[
e_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, e_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, e_3 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}
\] (74)

Also, the parameters required to analyze the stability using theorem (1) are:
\( C_{12} = (1 \ 0), C_{23} = (1 \ 0), F_{12} = (0 \ 1), F_{23} = (0 \ 1) \)
\( c_{12} = (1) c_{23} = (-1), f_{12} = (0 \ 1), f_{23} = (0 \ 1) \)

After the simulation, the appropriate control coefficients are obtained as follows. Also, the appropriate Lyapunov function for region \( X_2 \) is demonstrated in figure (4):
\[ \begin{align*}
K_1 &= (0.6882 \quad -2.2397) \quad (76) \\
K_2 &= (1.5510 \quad -3.1029) \quad (77) \\
K_3 &= (-3.5239 \quad -3.8100) \\
J_1 &= 0.0094 \quad (78) \\
J_2 &= 1.1873
\end{align*} \]

**Fig.6. Lyapunov function for region \( X_2 \)**

**VII. CONCLUSIONS**

In this paper, a class of hybrid systems that are able to model a wide range of practical systems is introduced and after providing the mathematical description and stability conditions of PWA systems in the form of LMIs, the upper bound of the cost function is calculated. In fact, theorem 4, theorem 5 and theorem 6 are the innovations of this article which prove that the problem of optimal control of PWA systems leads to BMI problem. Then, by minimizing the upper bound and using of GA and semi-definite programming, the controller coefficients are obtained. Note that we don’t use the GA for solving the optimization problem, in fact GA use for searching the acceptable solution. The importance of what is done lies in the fact that semi-definite programming is used to solve the optimization problem 1 and optimization problem 2 and this has less error than other methods. Considering the proposed method for optimal control is a comprehensive method, one can apply this method design optimal control for practical systems. In addition, in these two optimization problems because of the assumption of the continuity of the piecewise linear system, it lacks sliding mode which is a benefit of these design and makes these methods very suitable for designing optimal controllers for electronic power converters. In the end, we use two numerical examples to establish the effectiveness of the discussed methods. In two examples, simulation results show that optimization problem 1 that is based discontinuous quadratic lyapunov function is more efficient that one.

**REFERENCES**


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